Complex Geometry

Main reference: O'Well: Differential analysis on complex manifolds

Uhlenbeck-Yau: On the existence of Hermitian-Yang-Mills connections in stable vector bundle.

Goal:

Set: {points in S^2}

$\supsetneq$

Topological Manifold

$\supsetneq$

Riemann manifolds = Smooth manifolds + metric

$\supsetneq$

Complex manifolds ($\dim M = 2n$)

Kahler manifolds

$\{U_i\}_{i\in I}$ open cover

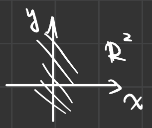
$\Phi_i \downarrow$

$\Phi_i(U_i) \subseteq \mathbb{R}^2$

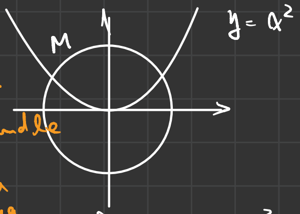
$\Phi_i(U_i\cap U_j) \longrightarrow \Phi_j(U_i\cap U_j)$
$\swarrow$

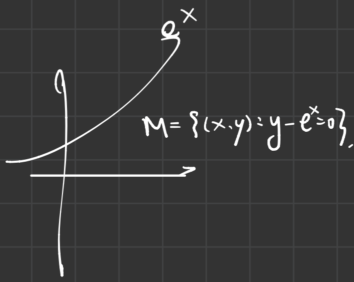
$\Phi_j \circ \Phi_i^{-1}$ is smooth, we say it's diffeomorphic

e.g. $\mathbb{C}^1 = \{ Z = x+\sqrt{-1}y : x.y \in \mathbb{R}^1 \}$



$(\sqrt{-1})^2 = -1$





Kahler manifolds + ample line bundle

$\supsetneq$ Kodaira embedding

Projective manifolds
(zero locus of complex mnfds)

$M = \{(x,y) : y - x^2 = 0\}$.

多项式零点表示的流形 (比较 easy)
poly nomlds

Notations: $Z \in \mathbb{C}$. $\boxed{\partial_z} f = \partial_x f - \sqrt{-1}\,\partial_y f$
$\quad\quad = \boxed{(\partial_x - \sqrt{-1}\,\partial_y)} f$

$\|$
$x+\sqrt{-1}y$

$\partial_{\bar z} f = (\partial_x + \sqrt{-1}\,\partial_y) f$

$\partial = \partial_z \otimes dz$, $\bar\partial = \partial_{\bar z} \otimes d\bar z$

$$d : C^{\infty}(X) \longrightarrow A^1(X)$$

光滑 1-form.

$$d = \frac{1}{2}(\partial + \bar{\partial}) = \partial_x \otimes dx + \partial_y \otimes dy$$

$$dz = dx + \sqrt{-1}\, dy. \quad d\bar{z} = dx - \sqrt{-1}\, dy$$

$$\frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \underset{\nearrow}{=} \boxed{dx \wedge dy}$$

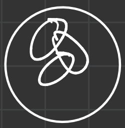
(Exercise) $\mathbb{R}^2$ Volume :

$\mathbb{C}^n$: Coordinate : $(z_1, z_2, \cdots z_n)$, $z_i = x_i + \sqrt{-1}\, y_i$

$$\bar{\partial} = \sum_{i=1}^{n} \partial_{\bar{z}_i} \otimes d\bar{z}_i$$

$$\partial = \sum_{i=1}^{n} \partial_{z_i} \otimes dz_i \qquad d = \partial + \bar{\partial}$$

Smooth manifolds : X



$\{U_i\}_{i \in \mathbb{N}}$ $\quad \mathbb{R}^n$ or B_1^n is homeomorphic

$$U_i \xrightarrow{\Phi_i} \Phi_i(U_i) \subseteq \mathbb{R}^n$$

①

②

$\Phi_j \circ \Phi_i^{-1} : \Phi_i(U_i \cap U_j) \downarrow \Phi_j(U_i \cap U_j)$ diffeomorphic

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Complex manifolds : X

① $\Phi_i(U_i) \subseteq \mathbb{C}^n$ is an open domain (e.g. $\Phi_i(U_i) \simeq B_r \subseteq \mathbb{C}^n$)

② $\Phi_j \circ \Phi_i^{-1} : \Phi_i(U_i \cap U_j) \longrightarrow \Phi_j(U_i \cap U_j)$ is biholomorphic.

Holomorphic functions and holomorphic morphism

(全纯)

$n = 1$; $U \subseteq \mathbb{C}^1$ is an open domain $\quad f \in C^1(U, \mathbb{C})$ is called holomorphic if ($u + \sqrt{-1}\, v$) it satisfies Cauchy - Riemann connection $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Leftrightarrow \partial_{\bar{z}} f = 0.$

$$\partial_{\bar{z}} f = (\partial_x + \sqrt{-1}\, \partial_y)(u + \sqrt{-1}\, v)$$
$$= (u_x - v_y) + \sqrt{-1}\,(u_y + v_x) = 0.$$

Let $U, V \subseteq \mathbb{C}^n$ be open domains.

A map $F : U \to V$ is called a holomorphic morphism. $F = (f_1(z), \cdots f_n(z))$, $z = (z_1, \cdots z_n)$

if $\forall f_i(z)$ is a holomorphic function for $i = 1, 2, \cdots n$.

biholomorphic = bijective + holomorphic

$\downarrow$

inverse is a holomorphic.

Def. X is called a Complex manifold if X is a smooth manifold and it satisfies ① ②

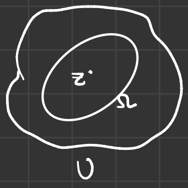
Why holomorphic? 会不会太强了？有什么被排在研究之外？

e.g. $F: \mathbb{C}^1 \longrightarrow \mathbb{C}^1$ $\underline{F(z)}$

$(x, y) \longmapsto (x, -y)$

「只带 $(z_1, \cdots, z_n)$ 参数地 而非 $(\bar{z}_1, \cdots, \bar{z}_n)$.」

$n=1$. $U \subseteq \mathbb{C}$. $\mathcal{O}(U) = \{ f \text{ is a holomorphic function on } U \}$.



$\partial\Omega$ is smooth.

$z \in \Omega$.

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi$$ 由边界信息决定.

Maximum principle: $\max\limits_{z \in \Omega} |f(z)|$ is obtained at $\partial\Omega$

「Remark: $f = u + \sqrt{-1}\, v$, $\underset{(\partial_x^2 + \partial_y^2)}{\overset{\shortparallel}{\Delta}} u = \Delta v = 0$ 」

Let $f \in C^\infty(\Omega)$

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{\pi} \underbrace{\int_{\Omega} \frac{\overset{= g}{\partial_{\bar{z}} f(\xi)}}{\xi - z} dx \wedge dy}_{\text{"不调合的部分"}}$$

$\Delta g = W = \partial_z \partial_{\bar{z}} f$

$g = \underline{\underline{G}} * W$ (基本解)

Green function

$n=2$ 时. $\log|x-y|$

$$\partial_z G = \frac{1}{\xi - z}$$

For $n \geq 2$: $\underset{\text{poly disk}}{\overset{\downarrow}{\mathbb{D}}} = \overbrace{\mathbb{D}_1 \times \mathbb{D}_1 \times \cdots \times \mathbb{D}_1}^{n} \subseteq \mathbb{C}^n$

$\mathbb{D}_1 = \{ z_i \in \mathbb{C} : |z_i| < 1 \}$.

$\underset{(z_1, z_2, \cdots, z_n)}{\overset{\shortparallel}{z}} \in \mathbb{D}$, $f \in \mathcal{O}(\mathbb{D})$

"Fubini"

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \overbrace{\int_{\partial D_1} \cdots \int_{\partial D_1}}^{n} \frac{f(\xi)}{(\xi_1 - z_1)(\xi_2 - z_2) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n.$$

$\int_{\partial D} \longleftarrow$

Let $g \in A^1(U)$ $U \subseteq \mathbb{C}^n$

$\partial_{\bar z} g = 0$ （必要条件）

Find $f \in C^\infty(U)$ s.t. $\partial_{\bar z} f = g$

$\searrow$ solve $\boxed{\bar\partial\text{-equation}}$ $\leftarrow$ Solvability depends on U

e.g. if $U = \underset{\text{polydisk}}{\mathbb{D}}$, $B_r(0) \subseteq \mathbb{C}^n$. Then $\bar\partial$-equation has a solution.

Hartog's principle:

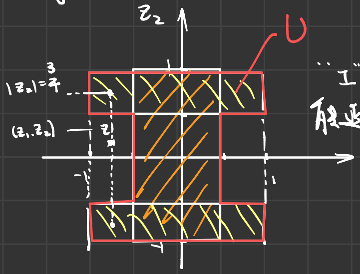
Let $n \geqslant 2$, $\bar U = \left\{ Z = (\underbrace{z_1, \cdots z_k}_{z'}, \underbrace{z_{k+1}, \cdots z_n}_{z''}) : \begin{array}{c} |z'| \leq \frac{1}{2} \text{ and } |z''| \leq 1 \\ \text{or} \\ \frac{1}{2} \leq |z'| \leq 1 \text{. and } \frac{1}{2} \leq |z''| \leq 1 \end{array} \right\}$.

where $0 < k < n$.

e.g. $n = 2$, $k = 1$ then there exists $F \in \mathcal{O}(\mathbb{D})$

$$\mathbb{D} = \{ 0 \leq |z_i| \leq 1 : i = 1, \cdots, n \}.$$

s.t. $F|_U = f$



“I” 能延拓成“口”

Sketch of the proof: $\underline{\text{curve}}$ $\mathcal{C} = \left\{ \xi = (\xi_1, \xi_2) : \begin{array}{l} \xi_1 = z_1 \\ |\xi_2| = \frac{3}{4} \end{array} \right\}$

(boundary)

circle. $F(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi$.

Riemann extension theorem:

Let $U \subseteq \mathbb{C}^n$ be an open domain.

Let $Z \subseteq U$ be a $\overset{}{\underset{\text{closed}}{\wedge}}$ smooth complex submanifold of U.

$f \in U \backslash Z$ $\downarrow$ closed.

Assume $\underset{\text{（余维数）}}{\operatorname{Codim}_{\mathbb{C}} Z \geqslant 2}$ $\left(\operatorname{Codim}_{\mathbb{R}} Z \geqslant 4 \right)$ ($\because n \geqslant 2$), Then:

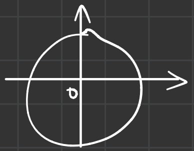
$\forall f \in \mathcal{O}(U \backslash Z)$, there $\exists F \in \mathcal{O}(U)$

s.t. $F|_{U \backslash Z} = f$

Proof. Exercise (Reduce to the setting of Hartog's principle)

Remark : ① $n = 1$. $\mathbb{C}^1 \supseteq U$

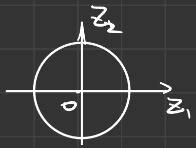
$\underline{\qquad\qquad}$
不成立



$U \setminus \{0\}$, $\mathrm{Codim}_{\mathbb{C}}(\{0\}) = 1$

$f = \frac{1}{z} \in \mathcal{O}(U \setminus \{0\})$

② $n = 2$.



$\mathrm{Codim}_{\mathbb{C}}(\{0\}) = 2$, $f \in \mathcal{O}(U \setminus \{0\})$

$$\downarrow$$

$F \in \mathcal{O}(U)$

Another way to define a complex manifold

(Real manifold + "symmetry")

e.g. $\mathbb{C}^1 = \{ z = x + \sqrt{-1}\, y : x, y \in \mathbb{R} \}$

$\overset{\shortparallel}{(x, y)} \longrightarrow (\mathbb{R}^2, J)$

$J_0 = \sqrt{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

$\cap$

linear transforms : $\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ $: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sqrt{-1}\, z = \sqrt{-1}\, x + y$

$(-y, x)$

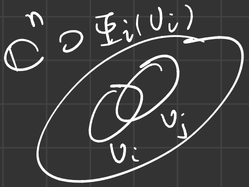
$(\mathbb{R}^2, J)$ s.t. $J \in GL_2(\mathbb{R})$ and $J \circ J = -1 = -Id$."

Exercise : $J = A^{-1} \circ J_0 \circ A^{-1}$, $A \in GL_2(\mathbb{R})$

We can choose a coordinate s.t. $(\mathbb{R}^2, J) \simeq \overset{\overset{\mathbb{C}^1}{\shortparallel}}{(\mathbb{R}^2, J_0)}$

$\left(A \begin{pmatrix} x \\ y \end{pmatrix} \right)$

Complex manifold   Recall:



$\Phi_j \circ \Phi_i^{-1}$ is biholomorphic ....

Definition: $J: \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is called a complex structure if it's a linear transform. and $J^2 = J \circ J = -1 = -Id$.

$$\mathbb{R}^{2n} = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$$

$$J: x_i \longrightarrow y_i. \qquad y_i \longrightarrow -x_i$$

$$z_i = x_i + \sqrt{-1}\, y_i \ . \quad J(z_i) = Jx_i + \sqrt{-1}\, Jy_i = -\sqrt{-1}(x_i + \sqrt{-1}\, y_i) = \boxed{-\sqrt{-1}}\, z_i$$

$$J(\bar{z}_i) = J(x_i - \sqrt{-1}\, y_i) = y_i + \sqrt{-1}\, x_i = \sqrt{-1}\,(x_i - \sqrt{-1}\, y_i) = \boxed{\sqrt{-1}}\, \bar{z}_i$$

$(z_1, \bar{z}_1, z_2, \bar{z}_2, \cdots, z_n, \bar{z}_n) \in \mathbb{R}^{2n} \otimes \mathbb{C}$ (denoted as $\mathbb{R}^{2n}_{\mathbb{C}}$)

$$\langle z_1, z_2, \cdots, z_n \rangle_{\mathbb{C}} = T^{1,0}(\mathbb{R}^{2n}_{\mathbb{C}})$$

$$\langle \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n \rangle_{\mathbb{C}} = T^{0,1}(\mathbb{R}^{2n}_{\mathbb{C}})$$

切空间 $\hookrightarrow T^1(\mathbb{R}^{2n}_{\mathbb{C}}) = T^{1,0}(\mathbb{R}^{2n}_{\mathbb{C}}) \oplus T^{0,1}(\mathbb{R}^{2n}_{\mathbb{C}})$

$\cong$

$\mathbb{R}^{2n}_{\mathbb{C}}$

$\updownarrow$ 1-1

J

$$J: T^{0,1} \longrightarrow T^{0,1} \qquad J: T^{1,0} \longrightarrow T^{1,0}$$

$$v \longmapsto \sqrt{-1}\, v \qquad v \longmapsto -\sqrt{-1}\, v.$$

$$\mathbb{C}^n = \langle z_1, z_2, \cdots, z_n \rangle_{\mathbb{C}} = T^{1,0}(\mathbb{R}^{2n}_{\mathbb{C}})$$

Let X be a $2n$-dimension smooth manifold
(Real)

An "almost" complex structure $J: TX \xrightarrow{C^\infty} TX$ $\forall x \in X$

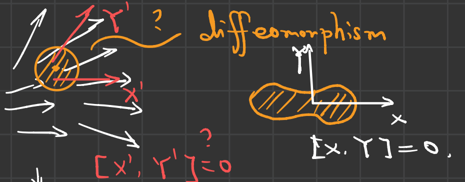
$\mathbb{R}^{2n} \simeq T_x X \xrightarrow{\text{linear}} T_x X$

and $J^2 = -1$

$T_x X = T_x^{1,0} X \oplus T_x^{0,1} X$

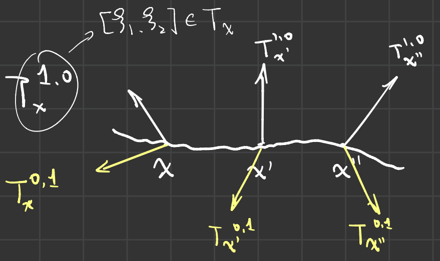
$T_x^{1,0} \overset{"\partial_z"}{=} \{ \xi - \sqrt{-1} \xi : \xi \in T_x X \}$

$T_x^{0,1} \overset{"\partial_{\bar z}"}{=} \{ \xi + \sqrt{-1} \xi : \xi \in T_x X \}$.



↓
smooth vector field

on $\mathbb{R}^2$



$\mathbb{C}^n : \langle \partial z_1, \cdots, \partial z_n \rangle$

$[\partial z_i, \partial z_j] \in \mathbb{C}^n$

necessary condition.

An almost complex J is called integrable if for $\forall x \in X$, for $\xi_1, \xi_2 \in T_x^{1,0}$, $[\xi_1, \xi_2] \in T_x^{1,0}$ (or equivalently $\xi_1, \xi_2 \in T_x^{0,1}$, $[\xi_1, \xi_2] \in T_x^{0,1}$).

$\underset{\Leftrightarrow}{\not\Rightarrow}$ $x \in U$, $U \xrightarrow{\Phi} \Phi(U) \subseteq \mathbb{C}^n$

Frobenius theorem: Let $E \subseteq TX$ be a distribution. Then for $\forall x \in X$, $\exists$ submanifold $\overset{Z}{\vee}$

$x \in Z \subseteq \underset{\substack{\downarrow \\ \text{open neighborhood} \\ \text{of } x}}{U} \subseteq X$ s.t. $E = TZ$

$\Updownarrow$

$\forall x' \in U$, $\forall \xi_1, \xi_2 \in E_{x'}$, $[\xi_1, \xi_2] \subseteq E$

Idea of proof

$\xi_1, \cdots, \xi_k$ be generator of E_x and extend over U

(Newlander - Nirenberg theorem)

(real analytic) smooth

Let J be an integrable almost complex structure on X.

Then X is a complex manifold.

$T_xX_{\mathbb{C}} \simeq T_x^{1,0} \oplus T_x^{0,1}$, $T^{1,0} \subseteq TX_{\mathbb{C}}$ is a distribution

$\downarrow$

4n - dim

$\Gamma\ x \in U.\quad TX_{\mathbb{C}} \quad U_{\mathbb{C}} \supseteq Z$

$\{\xi_1, \xi_2, \cdots \xi_n, J\xi_1, \cdots J\xi_n\}$

复空间

$\{\xi - \sqrt{-1}\,\xi\} = T^{1,0}X \simeq T_Z$ (with $T^{1,0}X \subseteq TX_{\mathbb{C}}$ written vertically)

$[\xi_1, \xi_2] \in T^{1,0}X \xrightarrow[\text{Frobenius}]{} Z \subseteq \boxed{U_{\mathbb{C}}}$ s.t.

integrable

$T^{1,0} \simeq TZ.$

$\xi_1, \cdots, \xi_n$

$< \xi_1 - \sqrt{-1}\,\xi_1, \cdots, \xi_n - \sqrt{-1}\,\xi_n >_{\mathbb{C}} \simeq T^{1,0}$

$(z_1, \cdots, z_n) \in Z \subseteq \mathbb{C}^n.$

用可积的近复结构得到 $\mathbb{C}^n$

Remark: In general case. $I = \bar{\partial} + \phi.\quad \phi: T^{0,1} \to T^{1,0}$

integrable $\Leftrightarrow$ $\boxed{\bar{\partial}\phi + \underline{\phi \wedge \phi} = 0}$

$\underline{\phi\wedge\phi} = \frac{1}{2}[\phi, \phi]$.

$\downarrow$

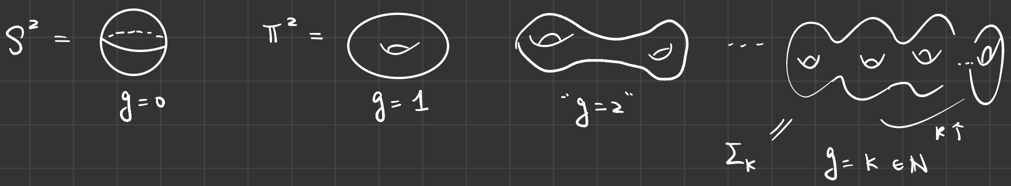
Maurer - Cartan equation.

A complex manifold is (X, J) . where X is a smooth manifold

J is integrable almost complex structure (which is called a complex structure)

Examples: (1) $\mathbb{C}^n$

(2) $n = 1$. X^2 is a 2-dim compact oriented smooth manifold.



For $\forall\ \Sigma_k$ $\exists$ a complex structure $\Rightarrow$ Σ_k is a 1-dim complex manifold.

(Jost) $\overline{}$ IS $n=1$

conformal structure

e.g. homogeneous polynomial f_{k+2} of degree $k+2$ on $\mathbb{C}^3$

$\dim_{\mathbb{C}}=1$ $\bigcirc\!\!\!\!X_{f_{k+2}}$ $\subseteq \mathbb{CP}^2$ $\qquad X_{f_{k+2}} \underset{\text{homeomorphic}}{\underline{\sim}} \Sigma_k$ for $k \in \mathbb{N}$

Riemann surface = algebraic curve

"线变为点"

(3) projective space

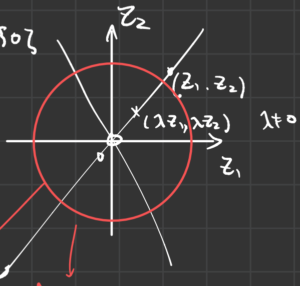
$\mathbb{C}^2 \setminus \{0\}$

$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$

for $\forall\ \lambda \neq 0 \in \mathbb{C}$

S^2

$\{x^2+y^2+z^2=1\}$.



$\mathbb{C}^2/\sim$

$\mathbb{CP}^1 \underline{\cong} S^2$

projective space.

$\dim_{\mathbb{C}}=1$

$\dim_{\mathbb{R}}=2$

Def: Projective space $\mathbb{CP}^n$

$\mathbb{C}^{n+1} \setminus \{0\} / \sim$

$(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)$

$\neq$
0

for $\forall \lambda \neq 0 \in \mathbb{C}$

$\mathbb{C}P^n$ has a homogeneous coordinate

$$[z_0, z_1, \cdots, z_n] = [\lambda z_1, \cdots \lambda z_n] \quad \text{for } \forall \lambda \neq 0$$

$$(z_1, \cdots z_n) \neq 0.$$

$U_i = \{ [z_0, \cdots z_n] \in \mathbb{C}P^n : z_i \neq 0 \}$ is an open set of $\mathbb{C}P^n$
$i = 0, 1, \cdots n.$

$\{U_0, U_1, \cdots U_n\}$ is an open cover. On U_i, we have

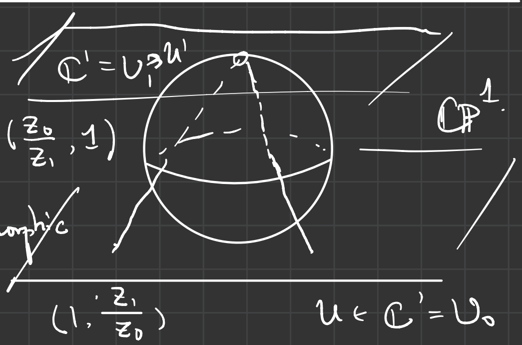
coordinate $(u_0^i, u_1^i, \cdots, u_n^i) = \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \cdots, \overset{i\text{th}}{\overset{\downarrow}{1}}, \cdots, \frac{z_n}{z_i}\right)$

$$\begin{tikzcd}
U_i \arrow[r, "{(u_0, u_1, \cdots u_{i-1}, \hat{u}_i, u_{i+1}, \cdots u_n)}", "\sim"'] \arrow[d] & \mathbb{C}^n \\
U_j &
\end{tikzcd}$$

$$\Phi_{ij} : \overset{\in U_i}{(u_0^i, u_1^i, \underset{\underset{u_k^i = \frac{z_k}{z_i}}{\downarrow}}{\ominus}, u_n^i)} \xrightarrow{\cdot \left(\frac{z_i}{z_j}\right)} (u_0^j, u_1^j, \underset{\underset{u_k^j = \frac{z_k}{z_j}}{\downarrow}}{\ominus}, u_n^j) \in U_j$$

$n = 1$, $\mathbb{C}P^1$. U_0, U_1

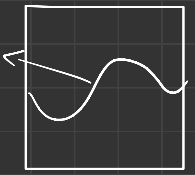
$S^2 \simeq$
homeomorphic

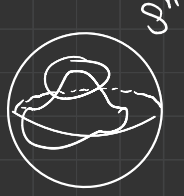


(4)

$\{f(x) = 0\}$

$f \in \mathbb{R}^n$





is a
$(n-1)$-dimensional
submanifold.

$0 \neq df|_{\{f(x)=0\}}$ 满秩

On $\mathbb{CP}^n$:

Let $f(z)$ be homogeneous polynomial of degree d

i.e. $\underline{f(\lambda z) = \lambda^d f(z)}$ $\forall\ 0 \neq \lambda \in \mathbb{C}$

z^d ✓ $\qquad z^d + 1$ ×

If $f(z) = 0$. then $f(\lambda z) = 0$. Zero locus of $f(z)$ on $\mathbb{CP}^n$

$$X_f = \{ \underset{(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)}{[z]} \in \mathbb{CP}^n : f(z) = 0 \}$$

X_f : is a smooth complex manifold of $\dim_{\mathbb{C}}(X) = n-1$

which is a submanifold in $\mathbb{CP}^n$.

$\forall$ Real $2n$ smooth manifold X, is $\underline{X \text{ is a complex manifold}}$?

$\Updownarrow$

$\exists$ complex structure J on X

X Couterexample :

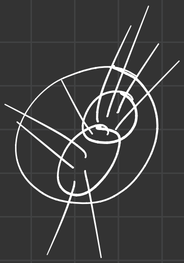
示性类计算: $\underline{S^{2n}}$. if $n \neq 1, 3$. Then there exists no complex structure on S^{2n}

<u>Open Question</u> S^6 $\exists J$?

complex manifold

* * *

Vector bundle



Let X be a topological manifold

isomorphic

A vector bundle of $\underline{\text{rank } r}$ $\pi : E \longrightarrow X$

consists of a topological manifold E

k-vector space

for $\forall x \in X$. $\pi^{-1}(x) \simeq k^r$ where k is a field either $\mathbb{R}$. or $\mathbb{C}$

and for $\forall x \in X$, $\exists$ an open set U. s.t. $\pi^{-1}(U) \underset{\text{homeomorphic}}{\simeq} U \times k^r$

Example : (1) trivial vector bundle

$$X \times k^r \xrightarrow{\pi} X$$

(2) $TX \xrightarrow{\pi} X$ $\qquad \pi^{-1}(U) = \{(x, \xi) : \xi \in T_x X,\ x \in U\}$

$\qquad\qquad\qquad\ \uparrow \qquad\qquad\qquad\qquad \simeq U \times \mathbb{R}^n$

n-dim mnfd.

(3) Given vector bundle $\begin{matrix} E \\ \downarrow \pi \\ X \end{matrix}$, then $\begin{matrix} E^* \\ \downarrow \\ X \end{matrix}$ is a vector bundle

$$F_x = \pi^{-1}(x) \simeq k^r$$

$$F_x^* := \operatorname{Hom}(F_x, k)$$

(4) T^*X : the dual of tangent vector bundle, which is called the <u>cotangent bundle</u>

smooth mnfd

$\downarrow$

$X^n \supseteq U \ni x$

$(x_1, \cdots, x_n)$ coordinate of U.

$T_x U = \langle \partial_{x_1}, \cdots, \partial_{x_n} \rangle_{\mathbb{R}}$ $\qquad dx_i(\partial x_j) = \delta_{ij}$

$T_x^* U = \langle dx_1, \cdots, dx_n \rangle_{\mathbb{R}}$

Def. Let X be a smooth manifold. Let $\pi : \begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a vector bundle.

A connection on E is $\nabla : \underset{\text{smooth section}}{A^0(E)} \longrightarrow \underset{\text{smooth 1-form}}{A^1(E)}$

and satisfies

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s \quad \text{where } f \text{ is a smooth func}, \ s \in A^0(E)$$

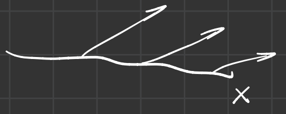
「 $x \in U$. 可定义 local.

作 cutoff function. χs. gobel 」

trivial vector bundle $\quad E = X \times k^r \xrightarrow{\pi} X$

$[\nabla, \nabla]$ 衡量纤维丛弯曲. 故 $F(\nabla) = [\nabla, \nabla]$

$\qquad\qquad\qquad\qquad\qquad\qquad \searrow$ curvature.



Lec 3

Vector bundles

Connections

Curvature

Hermitian manifolds

Kahler manifolds

(X, J) integrable : $[\xi, \xi] \in T^{0,1}$ for $\xi \in T^{0,1}$ $\Leftrightarrow$ Maurer-Carton equation

$d = \partial + \bar\partial$ $\Leftrightarrow$ $\bar\partial^2 = \bar\partial \circ \bar\partial = 0$ $\bar\partial \phi + \phi \wedge \phi = 0$

---

let X be a smooth manifold

$$\begin{tikzcd} \mathbb{R}^r \arrow[r] & E \arrow[d, "\pi :"'] \\ & X \end{tikzcd}$$

$\pi^{-1}(x) \simeq \mathbb{R}^r$ (or $\mathbb{C}^r$)

notation : $A^0(X, E) = \{$ smooth sections on $E \}$

$(A^0(U, E))$

$A^1(U, E) = A^1(U) \otimes A^0(U, E)$

$A^k(U, E) = A^k(U) \otimes A^0(U, E)$

(纤维丛下的导数)

Connections. $\nabla : A^0(E) \longrightarrow A^1(E)$ $\leftrightarrow$ $\nabla \in A^1(End(E))$, $f \in A^0(U)$, $S \in A^0(U, E)$

$$\nabla(fs) = df \otimes s + f \nabla s$$

Let $\{s_1, \cdots, s_r\}$ be a frame of E on U , $x \in U$. $S_1(x), \cdots S_r(x)$ basis of $\mathbb{R}^r \simeq \pi^{-1}(x)$

Denote $\nabla(S_i) = \boxed{A_{ij}} \otimes S_j$, $A = \begin{pmatrix} A_{11} & \cdots & \\ \vdots & \ddots & \\ & & A_{rr} \end{pmatrix}_{r \times r}$ $\in A^1(End(E)) = A^1 \otimes End(E)$

vertor field

one-form

e.g. , $E = X \times \mathbb{R}$ $A^0(E) = \{$ smooth functions on $X \}$

$\Gamma(E) : X \longrightarrow E$ $\nabla = d$ a connection.

graph of a function

$$\begin{tikzcd} A^0(X) \arrow[r, "\nabla = d"] & A^1(X) \arrow[r, "\nabla = d"] \arrow[d] & A^2(X) \arrow[d] \\ & {} & \vdots \end{tikzcd}$$

$\parallel$

$A^0(X, X \times R)$

$\parallel$

E

"正合列"

a complex

"$d^2 = 0$" $\Rightarrow$

define cohomology

$A^0(X, E) \xrightarrow{\nabla} A^1(X, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^n(X, E) \xrightarrow{\nabla} 0$, $n = \dim_{\mathbb{R}}(X)$

$\nabla \circ \nabla = \nabla^2 : A^0(X, E) \to A^2(X, E)$. i.e. $x \in U \subseteq X$. $\xi, \eta \in T(U)$,

$s \in A^0(U, X)$

$$\nabla^2(\xi, \eta)(s) = \underbrace{(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)}_{\overset{\parallel}{[\nabla_\xi, \nabla_\eta]}}(s) - \nabla_{[\xi, \eta]}(s)$$

Curvature of (E, ∇) : $F(\nabla) = \nabla \circ \nabla$: $A^0(E) \to A^2(E)$

(Obstraction for $A^0(E) \xrightarrow{\nabla} A^1(E) \xrightarrow{\nabla} A^2(E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^n(E) \to 0$ to be a complex

is the curvature $F(\nabla)$)

- - - - - - - - - -

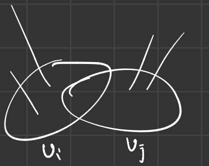
Another way to interpret $(E, \nabla, F(\nabla))$: by <u>associated principle bundle</u> of E

$GL(r,\mathbb{R}) \circlearrowright \mathbb{R}^r$ Def. A principle bundle is π $\begin{matrix} G \to P \\ \downarrow \\ E \end{matrix}$

which satisfies ① $\pi^{-1}(U) \simeq U \times G$. where $G = GL(r,\mathbb{R})$.

homeomorphic

② there exists a group action $G \circlearrowright P$ (the action $G \circlearrowright P$ Satisfies



$\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^n$

$U_i \cap U_j$ $\downarrow \Phi_{ij}$

$\pi^{-1}(U_j) \simeq U_j \times \mathbb{R}^n$

$x \in U_i \cap U_j$

$\Phi_{ij} : \underset{\mathbb{R}^n}{\underset{\|}{\pi^{-1}(x)}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\|}{\pi^{-1}(x)}}$

$\Phi_{ij} \in GL(r,\mathbb{R})$.

associativity

$$\begin{tikzcd}
\pi^{-1}(U) \times G \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & U \times G \times G \arrow[d] \\
\pi^{-1}(U) \arrow[r] & U \times G
\end{tikzcd}$$

$$\begin{tikzcd}
GL(r,\mathbb{R}) \circlearrowright \pi : & E \arrow[d] & \mathbb{R}^r \arrow[l] \\
& X &
\end{tikzcd}$$

associated with $\longleftrightarrow$ $\left(\begin{tikzcd} GL(r) \arrow[r] & P \arrow[d, "\pi"] \\ & X \end{tikzcd} \right)$ Principle bundle.

$$\begin{tikzcd}
\boxed{G} \arrow[r] \arrow[d] & P \arrow[r, phantom, "\ni"] \arrow[d, "\pi"] & p \\
T_{Id}G & X \arrow[r, phantom, "\ni"] & x
\end{tikzcd}$$

$\pi(p) = x$

$T_{Id} G \simeq$ Lie algebra of G

denoted as $\mathfrak{g}$

$$\begin{tikzcd}
TP \arrow[r, "\pi"] \arrow[d] & TX \\
\text{tangent bundle} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] & T_p P \arrow[r, "\pi"] & T_x X \arrow[l, dashed, bend left]
\end{tikzcd}$$

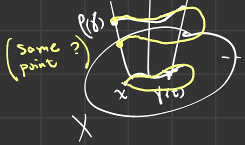
无限多种方式

A $\boxed{\text{connection}}$ on P is

a splitting : globally smooth defined

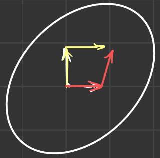
$$\begin{tikzcd}
TX \arrow[r, shift left, "\boxed{\rho}"] & TP \arrow[l, shift left, "\pi"]
\end{tikzcd}$$

$\Rightarrow TP \simeq TX \oplus \mathfrak{g}$ on U.



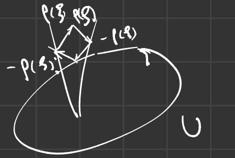
--> Horizontal lifting : $\rho(\frac{d}{dt}\gamma(t)) = \frac{d}{dt}\rho(\gamma)(t)$

有了 ρ. 方法唯一.



小邻域内不一定到同一点

衡量该差的量 Curvature.



Curvature on principle bundle (P, ρ)

connection.

For $\forall \xi, \zeta \in T(U)$. ($\overset{\ni x}{U}$ is a small nbhd)

$$[\rho(\xi), \rho(\zeta)] - \rho([\xi, \zeta]) = F(\xi, \zeta)$$

$$F : \Lambda^2 TX \longrightarrow \mathfrak{g}$$

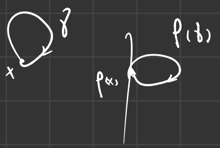
Proposition: Let $\pi \overset{P}{\underset{X}{\downarrow}}$ be a principle bundle with connection ρ

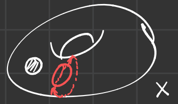
Then $F(\rho) = 0$ if and only if $\forall x \in X$. $\exists$ a small nbhd $x \in U \subset X$

for $\forall$ closed path $\gamma \subset U$: $\gamma : [0,1] \longrightarrow U$

$\gamma(0) = \gamma(1) = x$

its horizontal lifting $\rho(\gamma)$ is a closed path





red rig 不能缩成一点.

remark: Assume $F(\rho) = 0$

Then principle bundle $\overset{P}{\underset{X}{\downarrow}}$ is determined by a representation of $\pi_1(X)$

基本群.

$$P : \longleftarrow \begin{cases} (1)\ \text{curvature } F \\ (2)\ \text{representation of } \pi_1(X) \end{cases}$$

(E)

Let $\overset{E}{\underset{X}{\pi\downarrow}}$ be a vector bundle. A bundle metric $\psi = \langle\ ,\ \rangle$ on E

S^2表示可交换顺序 (s_1, s_2).

is in $C^\infty(X, S^2(E^*))$, when restricted on a fiber $\pi^{-1}(x)$.

$\psi : \underset{\mathbb{R}^r}{\underset{\text{IS}}{\pi^{-1}(x)}} \times \underset{\mathbb{R}^r}{\underset{\text{IS}}{\pi^{-1}(x)}} \longrightarrow \mathbb{R}$, is bilinear form. s.t.

ψ

(s_1, s_2)

$$\begin{cases} \langle s_1, s_2 \rangle = \langle s_2, s_1 \rangle \\ \langle s_1, s_1 \rangle \geq 0,\ \text{"="} \text{ iff } s_1 = 0 \end{cases}$$

Q: On a vector bundle $\overset{E}{\underset{X}{\pi\downarrow}}$, given $\begin{cases} ①\ \text{a connection } \nabla \\ ②\ \text{metric } \psi : \langle\ ,\ \rangle \end{cases}$

relation between ∇, $\langle\ ,\ \rangle$. ?

Def A connection ∇ is compatible with a metric $\psi : \langle\ ,\ \rangle$ if

for any local section s_1, s_2 (small nbhd U. $s_1, s_2 \in A^0(U, E)$)

$\forall$ vector field ξ.

$\langle \nabla_\xi s_1, s_2 \rangle + \langle s_1, \nabla_\xi s_2 \rangle \ =\!=\!=\ \xi(\langle s_1, s_2 \rangle)$, denoted. $\xi \langle\ ,\ \rangle = \langle \nabla_\xi\ ,\ \rangle + \langle\ ,\ \nabla_\xi \rangle$

Prop: Given a metric $\varphi = \langle\ ,\ \rangle$ on E. then there exists a connection ∇

infinitely many

that is compatible with ψ.

Pf: Exercise

Motivation

$x \in U \qquad \begin{matrix} T_X \\ \pi \downarrow \\ U \end{matrix} \simeq U \times \mathbb{R}^n \quad (\underset{\underset{\text{standard}}{\swarrow}}{\nabla_{std}} = d)$

∇ another connection on $\begin{matrix} T_X \\ \downarrow \\ U \end{matrix}$ by choosing a coordinate.

$$\begin{pmatrix} \text{differences} \\ \text{between } \nabla, \nabla_{std} \\ \text{on } U \end{pmatrix} = \begin{cases} 0^{th}\text{-order ; no difference} \\ 1^{st}\text{-order : } T_\nabla \\ 2^{nd}\text{-order : } F_\nabla \end{cases}$$

1^{st}-order difference:

$\xi, \xi, \eta \in \mathcal{T}(U)$ be vector fields
$= A_0(T_X)$

$\nabla_\xi : \mathcal{T}(U) \longrightarrow \mathcal{T}(U)$

$\underline{\nabla_\xi \eta \text{ is a vector field.}}$

stand case:

$\nabla_{std\,\xi}\, \eta = \partial_\xi \eta \;,\; \nabla_{std\,\eta}\, \xi = \partial_\eta \xi$

$\nabla_{std\,\xi}\, \eta - \nabla_{std\,\eta}\, \xi = [\xi, \eta]$

tortion of ∇ $\qquad T_\nabla(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - ([\xi, \eta])$.
挠率.

$$\left(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi\right)(\zeta) - \underbrace{\nabla_{[\xi,\eta]}(\zeta)}_{\downarrow \text{ normalization}}$$
$$\|$$
$$F_\nabla(\xi, \eta)(\zeta)$$

$$\left(\nabla_{std\,\xi} \nabla_{std\,\eta} - \nabla_{std\,\eta} \nabla_{std\,\xi}\right)(\zeta)$$
$$- \nabla_{std\,\underset{\underset{[\xi,\eta]}{\|}}{\underline{[\xi,\eta]}}}(\zeta) = 0$$

why

$$\underset{\underset{\partial_\xi}{\|}}{[\nabla_{std\,\xi}}, \underset{\underset{\partial_\eta}{\|}}{\nabla_{std\,\eta}}] = [\partial_\xi, \partial_\eta] = [\xi, \eta]$$

Theorem : For matric $g = \langle\, ,\, \rangle$ on $\begin{matrix} T_X \\ \pi \downarrow \\ X \end{matrix}$, there exists a unique connection ∇^{Lc} s.t $T_{\nabla^{Lc}} = 0$

(called Levi-Civita connection)

Pf : <u>Exercises</u>

Complex vector bundle.
$$\begin{tikzcd} E \arrow[d, "\pi"'] & \mathbb{C}^r \arrow[l, hook'] \\ X & \end{tikzcd}$$
, we can define <u>connection</u>

We can define a Hermitian metric $h: E \times E \longrightarrow \mathbb{C}$

which satisfies that on each fiber $\pi^{-1}(x)$,

$$\begin{tikzcd} h: \pi^{-1}(x) \arrow[d, phantom, "\cong"] & \times & \pi^{-1}(x) \arrow[d, phantom, "\cong"] \arrow[r] & \mathbb{C} \\ \mathbb{C}^r \arrow[d] & & \mathbb{C}^r \arrow[dll] & \\ (s_1, s_2, \cdots s_r) & & & \end{tikzcd}$$

can be written as

$$\left\{ \begin{aligned} & h(\Sigma a_i s_i, \Sigma b_i s_i) \\ & = \vec{a_i}^T (h_{ij})_{n \times n} \overline{b_j} \\ & = \sum_{i,j} a_i \cdot h_{ij} \cdot \overline{b_j} \end{aligned} \right.$$

$a_i, b_i \in \mathbb{C}$

and $(h_{ij}) = H$ is positive defined.

$\Longleftrightarrow$ $h(s,s) \geq 0$, "$=$" iff $s=0$.

example: $h: \mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{C}^1$ $\qquad z = x + \sqrt{-1}\, y$

$$dz \otimes \overline{dz} = (dx + \sqrt{-1}\, dy) \otimes (dx - \sqrt{-1}\, dy)$$
$$= \underbrace{(dx \otimes dx + dy \otimes dy)}_{\text{Euclidean metric}} - \sqrt{-1}\,(\underbrace{dx \otimes dy - dy \otimes dx}_{dx \wedge dy})$$

$$\begin{tikzcd} & T^{1.0}X \arrow[d, phantom, "\parallel"] & \\ \nabla^{LC} & (E, h) \arrow[d] & \\ & (X, J) \arrow[r] & \text{complex manifold.} \end{tikzcd}$$

is a complex vector bundle with Hermitian metric h. , ∇

$$T X_{\mathbb{C}} \simeq T^{1.0}X \oplus T^{0.1}X$$
$$d = \partial + \bar{\partial}$$
$$\nabla = \nabla^{1.0} + \boxed{\nabla^{0.1}} \quad \Rightarrow \text{implies cohomology}$$
$$\nabla^{0.1} = \bar{\partial}$$

$\downarrow$

Chern connection.

$\exists !\ \nabla^c$

$\nabla^{LC} = \nabla^c \to$ "kahler"

idea:

$$\left\{ \begin{aligned} & ①\ A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \longrightarrow 0 \\ & \quad A^0 \xrightarrow{\bar{\partial}} A^{0.1} \xrightarrow{\bar{\partial}} A^{0.2} \longrightarrow 0 \\ & \qquad \bar{\partial}^2 = 0. \\ & ②\ \nabla^{0.1}(\text{holomorphic}) = 0. \end{aligned} \right.$$

Let X be compact manifold, $\dim_{\mathbb{C}}(X) = n$, $\dim_{\mathbb{R}}(X) = 2n$

Def: A vector bundle $\pi : \begin{matrix} E \leftarrow \mathbb{C}^r \\ \downarrow \\ X \end{matrix}$ is called a holomorphic vector bundle if $\exists$ a cover $\{U_i, \Phi_i\}_{i \in I}^{X}$ ($\Phi_i : U_i \longrightarrow \mathbb{C}^n$)

$$\pi^{-1}(U_i) \underset{\text{biholomorphic}}{\overset{\sim}{\longrightarrow}} \Phi_i(U_i) \times \mathbb{C}^r$$

$\Longrightarrow$ $\exists$ holomorphic frame $\{S_1 \cdots S_r\}$ on U_i

$(\pi \begin{matrix} E \\ \downarrow \\ X \end{matrix}, h)$ $\longrightarrow$ Hermitian metric $h : E \times_X E \longrightarrow \mathbb{C}$ $\quad h_{ij} = h(S_i, S_j)$ $\quad (h_{ij})$ positive definite

$\alpha = \alpha^i S_i = \sum_i \alpha^i S_i \quad \alpha^i \in \mathbb{C}$

$\beta = \cdots$

$\Rightarrow h(\alpha, \beta) = \sum_{1 \le i, j \le n} h_{ij} \alpha^i \overline{\beta^j}$

$\Updownarrow$

$h(\alpha, \alpha) \geq 0$

"$=$" iff $\alpha = 0$

Let ∇ be a connection on E, ∇ is called "compatible with h" if

$$d(h(\alpha, \beta)) = h(\nabla\alpha, \beta) + h(\alpha, \nabla\beta)$$

Definition: A connection ∇ on a Hermitian vector bundle (E, h) is called a Chern-connection if $\begin{cases} (1) & \nabla \text{ is compatible with "}h\text{"} \\ (2) & \nabla^{0,1} = \bar{\partial} \end{cases}$

$(TX)_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$

$\nabla = \nabla^{1,0} + \nabla^{0,1}$

if $\xi \in T^{1,0}$, $\nabla_\xi^{0,1} = 0$.

$(\xi \in T^{0,1} \quad \nabla_\xi^{1,0} = 0)$

Prop: For $\forall$ Hermitian bundle $\pi \begin{matrix} (E,h) \\ \downarrow \\ X \end{matrix}$, $\exists!$ Chern-connection ∇^C associated with (E, h)

Sketch of proof: Choose a holomorphic frame $\{S_1 \cdots S_r\}$, $\nabla^C = \overset{\partial + \bar{\partial}}{\overset{\parallel}{d}} + A$

$(\nabla^C)^{0,1} = \bar{\partial} \Leftrightarrow (A)^{0,1} = 0$ $\;\vdots\;$ $\partial_\alpha(h(S_i, S_j)) = (d_\alpha h(S_i, S_j))^{1,0} = (h(\nabla_\alpha S_i, S_j) + h(S_i, \nabla_\alpha S_j))^{1,0}$

$$= h(\nabla_\alpha^{1,0} S_i, S_j) + \underline{h(S_i + \underset{\underset{\bar{\partial}(S_j)}{\parallel}}{\boxed{\nabla_\alpha^{0,1} S_j}})} = 0$$

$(\nabla_\alpha^C)^{1,0} S_i = (A^{1,0})_{\alpha i}^{k} S_k$

$\alpha \in T^{1,0}$ $\quad \sum_{1 \le k \le r} (A^{1,0})_{\alpha i}^{k} S_k$

$$\partial_\alpha h(S_i, S_j) = (A^{1,0})_{\alpha i}^{k} h(S_k, S_j)$$

$$(A^{1,0})_{\alpha i}^{k} = \sum_{1 \le j \le r} (h^{-1}_{kj}) \partial_\alpha h(S_i, S_j)$$

$$= h^{kj} \partial_\alpha h_{ij}$$

$\Longleftrightarrow \quad (\nabla^C)^{1,0} = h^{-1} \partial h$ $\qquad \square$

(E, h) : Chern connection $\quad \nabla^C = (\nabla^C)^{1,0} + (\nabla^C)^{0,1}$

$$= \partial + h^{-1}\partial h + \bar{\partial}$$

$$= d + h^{-1}\partial h.$$

A special case $\mathbb{C}^n \longrightarrow T^{1,0}X$ is a holomorphic vector bundle

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & T^{1,0}X \arrow[d, "\pi"'] \\ & X \end{tikzcd}$$

$\mathrm{rank}(T^{1,0}X) = n$.

$(\mathbb{C}, J)$ — (x, y), $y \overset{\shortparallel}{=} Jx$

$$\mathbb{C}^1: \quad dz\otimes d\bar z = \left(\boxed{dx\otimes dx + \underbrace{dy\otimes dy}_{J dx\otimes J dx}}\right) + \underbrace{\sqrt{-1}\,(dy\otimes dx - dx\otimes dy)}_{-2\sqrt{-1}\,dx\wedge dy \,=\, -dz\wedge d\bar z}$$

Euclidean metric on $\mathbb{R}^2$ Symplectic form

Let h be a Hermitian metric on $T^{1,0}X$

Let $\xi', \eta' \in TX$ be real vector field, $\underset{\xi}{\xi' - \sqrt{-1}J\xi'}$, $\underset{\eta}{\eta' - \sqrt{-1}J\eta'} \in T^{1,0}X$

$$h(\xi,\eta) = \mathrm{Re}\,(h(\xi,\eta)) + \sqrt{-1}\,\mathrm{Im}\,(h(\xi,\eta))$$

Define $g(\xi',\eta') = \frac{1}{2}\mathrm{Re}\,(h(\xi,\eta))$ is a Riemann metric on X

Define $\omega(\xi',\eta') = -\frac{1}{2}\mathrm{Im}\,(h(\xi,\eta))$ be a 2-form

By definition, $\underline{g(\xi',\eta') = \omega(\xi', J\eta')}$, $\underline{g(J\xi', J\eta') = g(\xi',\eta')}$ $\Longrightarrow$ $\omega(J\xi',\eta') \overset{\shortparallel}{=} g(\xi',\eta')$

e.g. $\omega = \dfrac{-2\,dx\wedge dy}{2} = dx\wedge dy = dx\wedge(J dx)$

$g = \frac{1}{2}(dx\otimes dx + dy\otimes dy)$, $g(\cdot, J\cdot) = \frac{1}{2}(dx\otimes J dx + dy\otimes \overset{-dx}{J dy})$

$= \frac{1}{2}(dx\otimes J dx - J dx\otimes dx)$

$= dx\wedge J dx$

For $\left(\begin{tikzcd} T^{1,0}X \arrow[d, "\pi"'] \\ X \end{tikzcd}, h\right)$ $h = 2g - 2\sqrt{-1}\,\omega$ $\{\omega(J\cdot,\cdot) = g(\cdot,\cdot)$ defined on $TX\}$

$T^{1,0}X \subseteq (TX)_{\mathbb{C}}$

$h: \longrightarrow \nabla^{c}$ $\longrightarrow$ defined on $TX \ni \xi', \eta'$

$h(\xi,\eta)$

$g \longrightarrow \nabla^{LC}$

A complex manifold X is called a Kahler manifold if $\exists$ a Hermitian metric h on $T^{1,0}X$ s.t. $\nabla^{c} = \nabla^{LC}$, then $\left\{\begin{array}{l} g \text{ is called a Kahler metric} \\ \omega \text{ is --- Kahler form} \end{array}\right\}$.

$h \overset{\shortparallel}{=} 2g - 2\sqrt{-1}\,\omega$

Proposition: The following statements are equivalent:

(1) g is a Kahler metric $(\nabla^c = \nabla^{LC})$

(2) $\nabla^{LC} J = 0$

(3) ω is d-closed.

(4) For any $x \in X$, $\exists$ small nbhd $B_r(x)$, we can pick a holomorphic coordinate s.t. $\omega = \frac{\sqrt{-1}}{2}\left(\sum_{1\le i,j\le n} \delta_{ij} + O(r^2)\right) dz_i \wedge d\bar{z}_j$

$g_{ij} = \delta_{ij} + O(r^2)$

(5) For $\forall x \in X$. $\exists$ a small nbhd U s.t. on U. $\omega = \sqrt{-1}\,\partial\bar{\partial}\varphi$, $\varphi \in C^\infty(U, \mathbb{R})$

Proof: "(1) $\Rightarrow$ (2)" $\nabla^c = \nabla^{LC}$. Choose a unitary holomorphic frame $\{s_1 \cdots s_n\}$ s.t. h_{ij} = constant.

$$\nabla^C h = 0. \quad 0 = \nabla^{LC} h = \nabla^{LC}(2g - 2\sqrt{-1}\,\omega)$$
$$= 0 - 2\sqrt{-1}\,\nabla^{LC}\omega \quad g(J;\cdot)$$
$$= -2\sqrt{-1}\, g(\nabla^{LC} J \cdot\,\cdot)$$
$$= 0 \quad \Leftrightarrow \quad \nabla^{LC} J = 0 \quad \checkmark$$

"(2) $\Rightarrow$ (1)" Let $\xi', \eta' \in TX$, $\xi = \xi' + \sqrt{-1}J\xi'$, $\eta = \eta' - \sqrt{-1}J\eta' \in T^{1,0}X$

(a) $(\mathcal{L}_{\eta'} J)(\xi) = [\eta', J\xi] - J[\eta', \xi]$

Lie derivative.

$$\overset{\text{torsion free}}{=} \underline{\nabla^{LC}_{\eta'}(J\xi)} - \nabla^{LC}_{J\xi}(\eta') - J\left(\nabla^{LC}_{\eta'}\xi + \underline{\nabla^{LC}_{\xi}\eta'}\right)$$

$J\nabla^{LC}_{\eta'}(\xi)$

$$= J\left(\nabla^{LC}_{\xi}\eta' + J\nabla^{LC}_{J\xi}\eta'\right)$$

$\nabla^{LC}_{\xi'}\eta' + \sqrt{-1}\,\nabla^{LC}_{J\xi'}\eta'$

(b) By the same computation (replace ∇^{LC} by d) we get

$$-J \circ (\mathcal{L}_{\eta'} J)(\xi) = \bar{\partial}_\xi \eta$$

0.1-部分

$$\Rightarrow \bar{\partial}_\xi \eta = \nabla^{LC}_\xi \eta' + J \boxed{\nabla^{LC}_{J\xi}}\eta' = \nabla^{LC}_\xi\eta' - \sqrt{-1}\,\underline{J\nabla^{LC}_\xi\eta'} = \nabla^{LC}_\xi(\eta' - \sqrt{-1}\,\eta')$$

$\nabla^{LC}_{J\xi} = -\sqrt{-1}\,\xi$.

$J\nabla^{LC}_\xi\eta' = \nabla^{LC}_\xi(J\eta')$

$\eta' - \sqrt{-1}\,\eta' = \eta$, $\nabla^{LC}_\xi(\eta' - \sqrt{-1}\,\eta') = \nabla^{LC}_\xi \eta$

(2) $\Leftrightarrow$ (3) ($\Leftarrow$) Lemma. Let ξ_1, ξ_2, ξ_3 vector fields in TX, commute with each other (e.g. $[\xi_i, \xi_j] = 0$)

then (a) $d\omega(\xi_1, \xi_2, \xi_3) = \langle (\nabla^{LC}_{\xi_1} J)\xi_2, \xi_3 \rangle_g + \langle (\nabla_{\xi_2} J)\xi_3, \xi_1 \rangle_g + \langle (\nabla_{\xi_3} J)\xi_1, \xi_2 \rangle_g$

(b) $2\langle (\nabla_{\xi_1} J)\xi_2, \xi_3\rangle_g = d\omega(\xi_1, \xi_2, \xi_3) - d\omega(\xi_1, J\xi_2, J\xi_3)$

$\uparrow$ Koszul formula for ∇^{Lc}

Exercise

(4) => (3) $\quad d\omega(x) = \underset{0}{d\left(\sum_{i,j} dz_i \wedge d\bar z_j\right)} + \lim_{r\to 0} O(r) = 0$

(3) => (4) $\quad d\omega = 0$, let $\{z_1, \dots, z_n\}$ be local. holo coordinate near $x=0$, $\partial_{z_1}, \dots \partial_{z_n}$.

$$\omega = \sum_{1\le i,j\le n} \frac{\sqrt{-1}}{2}\left(\delta_{ij} + \sum a_{kij} z_k + \sum b_{kij} \bar z_k + O(|z|^2)\right) dz_i \wedge d\bar z_j$$

$d\omega = 0 \Rightarrow a_{kij} = a_{ikj}$ $\quad b_{kij} = \overline{a_{kij}}$ ($\because \omega = \bar\omega$)

choose new coordinate $\quad z_k = w_k + \frac{1}{2}\sum_{i,j} C_{kij} w_i w_j$ $\quad C_{jki} = -a_{kij}$

$$\omega = \sum \frac{\sqrt{-1}}{2}\left(\delta_{ij} + o(|w|^2)\right) dw \wedge d\bar w$$

$\{w_1, \dots, w_n\}$ coordinate.

(5) ⇒ (3) $\quad d\omega = d(\sqrt{-1}\partial\bar\partial\varphi) = 0$, $\quad d = (\partial + \bar\partial)$, $\quad \begin{cases}\partial^2 = 0 \\ \bar\partial^2 = 0\end{cases}$

(3) => (5). $\quad d\omega = 0 \Rightarrow \exists$ one form η s.t. $d\eta = \omega$

locally near x

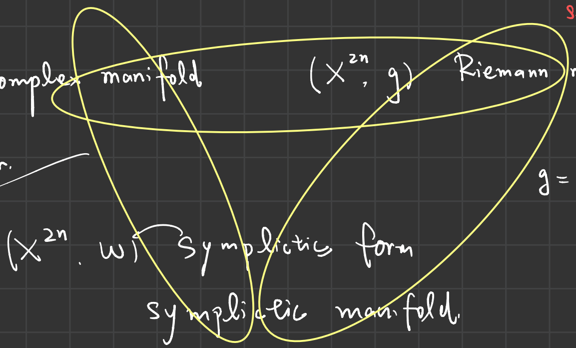
$\eta^{1,0} + \eta^{0,1} \overset{?}{\sim} \sqrt{-1}\partial\bar\partial\varphi$

$\omega = \bar\omega \Rightarrow \eta^{1,0} = \overline{\eta^{0,1}}$ > ✓

(solve →) $\bar\partial\varphi = \eta^{0,1}$

$\bar\partial\varphi = \eta^{0,1}$ where $\bar\partial(\eta^{0,1}) = 0 \xrightarrow[L^2\text{-estimate}]{\text{Hormander's}} \exists\varphi$. □

$\forall, \bar\partial\varphi$, ⇒ Kahler



Comments: (X^{2n}, J) Complex manifold $\quad (X^{2n}, g)$ Riemann manifold

s.t. $g(J., J) = g(\,)$ & $g(J.\cdot)$ is closed.

$\omega(\_, J\_)$ is Riemann metric

Kahler.

$g = \omega(\_, J\_)$

(X^{2n}, ω) Symplictic form

symplictic manifold.

e.g. (1) $\mathbb{P}^n$ is Kahler manifold   homogeneous coordinate $[z_0, \cdots z_n]$

Fubini - Study metric (form) $U_i = \{ z_i \neq 0 \}$

On U_i, $\omega_{FS} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \left(|\frac{z_0}{z_i}|^2 + \cdots + |\frac{z_n}{z_i}|^2 \right)$

kahler metric.

(2) $\forall$ complex submanifold $X \subseteq \mathbb{P}^n$. X is Kahler. $\omega = \omega_{FS}|_X$

(3) NOT $\forall$ complex mnfd is Kahler

<u>Hopf surface</u> $\underset{\text{diffeomorphism}}{\simeq} S^1 \times S^3$ $h^1(S^1 \times S^3, \mathbb{R}) = 1$ ← odd.

↓

complex manifold.

Kahler X. $h^1(X, \mathbb{Z}) = 0 \ (\mathbb{Z}_2)$ ↘ even   ∴ It's not kahler mfd.

Sheaf theory

(1) Category

(2) sheaf on topological space.

(3) struture sheaf of complex manifold

---

functor between categories

$F: \mathcal{C} \longrightarrow \mathcal{D}$ $c \in Obj(\mathcal{C})$, $F(c) \in Obj(\mathcal{D})$

$c \xrightarrow{f} d \in Mor(\mathcal{C})$

$F(c) \xrightarrow{F(f)} F(d) \in Mor(\mathcal{D})$

$\lim_{i \in N} a_i \longrightarrow a$

Denote I as a (small) category (which says the role of index)

Consider functor $F: I \longrightarrow \mathcal{C}$

$$\begin{tikzcd} I & a \arrow[r, "f"] & b & \cdot c \end{tikzcd}$$

<u>limit of $F: I \longrightarrow \mathcal{C}$</u> is an object s.t.

$$\begin{tikzcd} & \lim(F) \arrow[dl] \arrow[d] \arrow[dr] & \\ F(a) \arrow[r, "F(f)"] & F(b) & F(c) \end{tikzcd}$$

Category : $\mathcal{C}$ which consists of

objects : $c \in Obj(\mathcal{C})$ ← a set

morphisms: $(f: c \longrightarrow d) \in Mor(\mathcal{C})$ ← a set

---

example :

$V(\mathbb{R}, n)$: n-dim vector space $/ \mathbb{R}$

(1) as a set $\left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, v_i \in \mathbb{R} \right\}$

(2) $\left(V(\mathbb{R}, n), f \in GL(n, \mathbb{R}) \right)$. $f: V \longrightarrow V$

$Mor(v_1, v_2) = \{ f \in GL(n, \mathbb{R}), f(v_1) = f(v_2) \}$

(3) Top $= \{$ topological space. continuous maps $\}$

Ab : (Abelian groups, group morphisms)

Rings (Rings rng ~)

R-modules   :   :

<u>colimit of F</u> is an object. $colim(F) \in obj(\mathcal{C})$

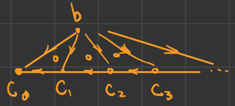
$$\begin{tikzcd} & colim(F) & \\ F(a) \arrow[ur] \arrow[r, "F(f)"] & F(b) \arrow[u] & F(c) \arrow[ul] \end{tikzcd}$$

example (1) limit (inverse limit) , $I = (\mathbb{N}, \ \leftarrow\!\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet \cdots)$ with points labelled 0 1 2 3 4

$a_i \in \mathbb{R}$, a convergent series $b = \sum_{i \in \mathbb{N}} a_i$ $\left(C_n = \sum_{1 \le i \le n} a_i \ \ C_{n+1} \longrightarrow C_n \right) \in \mathcal{C}$ $\in \mathbb{R}$

then b is the limit of $\{ \sum_{0 \le i \le n} a_i \}$

$\sum_{i=0}^{\infty} a_i \ //$



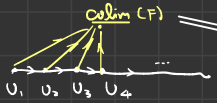
$I \xrightarrow{F} \mathcal{C}$

$n \longmapsto C_n$

(2) colimit (direct limit) $I = (\bullet\!\to\!\bullet\!\to\!\bullet\!\to\!\bullet\!\to\!\bullet \cdots)$

$\mathcal{C} = \{$ open subsets of a topological space X, $U \underset{\text{inclusion}}{\hookrightarrow} V \}$

$F : I \longrightarrow \mathcal{C}$. $U_i = F(i)$, $i \in \mathbb{N}$



$\text{colim}(F) =\!=\!=$ union of $U_i = \bigcup_{j \in \mathbb{N}} (U_i)$.

Def A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called $\underset{\underline{\text{left-exact}}}{\underline{\text{right-exact}}}$ if it preserves finite $\underset{\text{colimits}}{\text{limits}}$.

i.e. $F(\lim \mathcal{C}_i) = \lim F(\mathcal{C}_i)$

$F(\text{colim } \mathcal{C}_i) = \text{colim} F(\mathcal{C}_i)$

Functor . $F : \mathcal{C} \longrightarrow \mathcal{D}$ $G : \mathcal{D} \longrightarrow \mathcal{C}$.

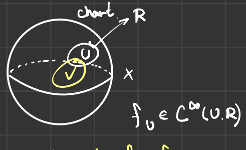
<u>Hom-set adjunction</u> F is left adjoint to G denoted as : $F \dashv G$

(G ... right ... ... F)

$\text{Hom}_{\mathcal{D}}(F(c), d) \simeq \text{Hom}_{\mathcal{C}}(c, G(d))$

Prop : Consider functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ assume $F \dashv G$

Then F is right exact . G is left exact .



$f_U \in C^{\infty}(U, R)$

$U \cap V : f_U = f_V$

$\Longrightarrow$ global function

$f = \{ f_U \}$.

Let X be a topological space.

$\mathcal{C}$ be a Category (Sets. Ab, R-mods, Rngs).

A <u>presheaf</u> of $\mathcal{C}$ on X : F is given by :

(1) $\forall$ open subset $U \subseteq X$, $F_U : U \longrightarrow \mathcal{C}$

(2) $\forall \ \underset{\underset{\mathcal{C}}{\downarrow}}{V} \subset \underset{\underset{\mathcal{C}}{\downarrow}}{U}$ $r_V^U : V \longrightarrow U$. $F_U \circ r_V^U = F_V$

Better ✓ (2)' $\forall \ V \subset U$, define $r_V^U : F_U \longrightarrow F_V$, and it satisfies

$W \subseteq V \subseteq U$

$r_W^U F_U = r_W^V \circ r_V^U F_U$

Ex (1) example of presheaf

Let X be a topological space. let U, V be open subsets of X.

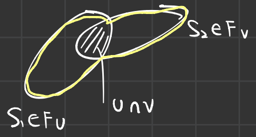
s.t $U \neq X$. $V \neq X$. $U \cup V = X$

U, $F_U = \{ s \in \mathcal{C}^{\infty}(U) \cap L^{\infty}(U) \}$

V , $F_V = \{ s \in \mathcal{C}^{\infty}(U) \cap L^{\infty}(U) \}$

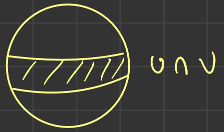
For $W \subseteq U$. or $W \subseteq V$. $F(W) = \{ s \in C^{\infty}(U) \cap L^{\infty}(U) \}$. $F : X \longrightarrow R$

for $W \nsubseteq U$, and $W \nsubseteq V$ $F(W) = \phi$. Then We have a presheaf



$1 \equiv S_1 \in F(U)$

$1 \equiv S_2 \in F(V)$



$$r^U_{U\cap V}(S_1) = r^V_{U\cap V}(S_2)$$

✻ ? not work for presheaf

$S \in F_{U\cup V}$ s.t. $\begin{cases} r^{U\cup V}_U(S) = S_1 \\ r^{U\cup V}_V(S) = S_2 \end{cases}$

$$r^U_{U\cap V}(S_1) = r^V_{U\cap V}(S_2) = 1$$

But $U \cup V \not\subset U$, $\not\subset V$ $\therefore \not\exists$. 不存在 s

---

A sheaf F on X is a pre-sheaf and satisfies for any open set U and a cover $\{U_i\}$ of U

$\forall$ a collection $S_i \in F_{U_i}$ and $S_i|_{U_i\cap U_j} = S_j|_{U_i \cap U_j}$

then there exists $S \in F(U)$ s.t $S|_{U_i} = S_i$

Sheafification

Let F be a presheaf on X. let $U \subseteq X$ an open subset

$$H^0(U, F) = \{ \textstyle\prod_i S_i \in \prod_i \overset{=F_{U_i}}{F(U_i)} : \{U_i\} \text{ is a cover of } U$$
on $U_i \cap U_j$. S_i is locally compatible with S_j $\}$.

" $S_i|_{U_i\cap U_j} = S_j|_{U_i\cap U_j}$

sheafification of F: $F^{\#}$

$F^{\#} = \underset{\overleftarrow{U}}{\operatorname{colim}}\, H^0(U, F)$ is a sheaf

粘起来 (colim)

Remark: Category: $$\begin{tikzcd} Sh_X \arrow[r, shift left, "i"] & PreSh_X \arrow[l, shift left, "\#"] \end{tikzcd}$$

i: forgetful functor

$\# \dashv i$ (adjoint each other)

Sheaf. $F: \text{Open}_X^{op} \longrightarrow \mathcal{C}$ Abelian group, R-module

(1) tensor $\dashv$ Hom

(2) direct image, inverse image π: $$\begin{tikzcd} \mathcal{F} \arrow[d] & \mathcal{G} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(3) Kernel, cokernel.

$\forall$ bilinear

$$\begin{tikzcd}
 & \langle A, B \rangle \arrow[dl] \arrow[dr] \arrow[d, dashed, "\exists"'] & \\
A & A \otimes B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

A, B be $\underset{\downarrow \\ \text{ring}}{\textcircled{R}}$- modules

$\underline{\underline{A \otimes_R B}}$ as a colimit

$A. B. C.$ R- modules

$$\mathrm{Hom}_R(A \otimes_R B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$$

$$\otimes_R \dashv \mathrm{Hom}_R$$

$\Big) \longrightarrow$ tensor. hom. adjunction

For sheaves $\mathcal{F}. \mathcal{G}. \mathcal{H}.$ on X of R- modules

$$\mathcal{F} \otimes \mathcal{G} \quad . \quad \mathcal{H}om_X(\mathcal{F}. \mathcal{G})$$

$$\mathrm{Hom}_X(\mathcal{F} \otimes \mathcal{G}. \mathcal{H}) \simeq \mathrm{Hom}_X(\mathcal{F}. \mathcal{H}om$$

direct image . inverse image

F: sheaf on X

G: sheaf on Y

$\pi : X \to Y$ continuous map

inverse image: $\pi^* G$, sheaf on X

$\forall$ open subset $U \subseteq X$

$$\pi^* G(U) = \underbrace{\Big(\operatorname*{colim}_{\substack{\pi(U) \subseteq V \\ \text{open}}} G(V) \Big)}_{\text{presheaf}}{}^{\#}$$

direct image $\pi_* F$ sheaf on Y

$\forall$ open set $V \subseteq Y$

$$\pi_* F(V) = F(\pi^{-1} V)$$

direct image , inverse image. adjunction

$$\mathrm{Hom}_Y(\pi_* F, G) \simeq \mathrm{Hom}_X(F, \pi^* G)$$

$$\boxed{\pi_* \dashv \pi^*}$$

Ex. (1) $\pi : \begin{matrix} F \\ \downarrow \\ X \end{matrix} \longrightarrow \{pt\}^{G = Ab}$ $\qquad \pi_* F = \{\text{global sections of } F\}$

$\qquad = \Gamma(F)$

$\pi^* G$: a constant sheaf

(2) $i_x : \{pt\} \longrightarrow \begin{matrix} F \\ \downarrow \\ X \end{matrix}$

$i_x(pt) = x$

denoted as $\mathcal{F}_x$ $\qquad$ stalk functor

$$i_x^* F = \Big(\operatorname*{colim}_{\substack{x \in V \\ \text{open}}} F(V) \Big)^{\#} \ : \text{stalk of } F \text{ at } x$$

$$i_{x*}(G)(V) = \begin{cases} x \in V : & Ab \\ x \notin V : & 0 \end{cases}$$

"skyscraper functor"

Let F, G be sheaves of Ab on X

sheaf morphism $f : F \to G$

$\mathrm{Ker}(f)$, $\mathrm{coker}(f)$ are sheaves on X

$f : F \to G$. by $\underbrace{\text{stalk}}_{\text{direct limit}}$ functor $f_x : F_x \to G_x$

$(\underbrace{\mathrm{Ker}}_{\text{finite limit}}(f))_x = \mathrm{Ker}(f_x)$

and $(\mathrm{coker}(f))_x = \mathrm{coker}(f_x)$.

direct limit commutes with finite limit.

Cor: Given $f: F \to G$. Then f is isomorphism iff $f_x : F_x \longrightarrow G_x$ is isomorphism for $\forall x \in X$

<u>Proof</u> (idea) $\ker(f) \equiv 0$

$\Leftrightarrow (\ker(f))_x = 0 \quad \forall x \in X$

$\Leftrightarrow \ker(f_x) = 0.$

Let X be a complex manifold

Sheaf $\mathcal{O}_X$ is called the <u>structure sheaf</u> on X.

$\forall$ open set $U \subseteq X$. $\mathcal{O}_X(U) = \{ f \text{ is a holomorphic fn on } U \}$

Proposition: For $\forall x \in X$ ← complex manifold $\quad \mathcal{O}_{X,x}$ is <u>Noetherian</u> <u>local</u> ring

$\Rightarrow$ $\boxed{\text{Cor}}$: An analytic subset Y of X is locally : $\bigcap_{1 \le i \le m} Z_i(f_i)$ ($\because$ Noetherian)

<u>Def</u> An analytic subset of $U \subseteq \mathbb{C}^n$ is the intersection of Zero locus of holo function

" $= \bigcap_{i \in I} Z_i(f_i)$ "

Let X be complex manifold

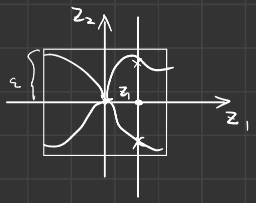
Sheaf : $\underset{\substack{\downarrow \\ \text{functor}}}{\mathcal{F}}$: $\text{Open}_X^{op} \longrightarrow$ Category : Abelian group (or ring)

$(X, \mathcal{O}_X) \rightarrow$ structure sheaf (holomorphic function)

$\forall x \in X$. $\mathcal{O}_{X_x}$ is Noetherian local ring

Sketch of pf: Denote $\mathcal{O}_{X,x} \overset{by}{=} \mathcal{O}_{\textcircled{n}} \leftarrow$ dimension $\qquad n=2$

$0 \not\equiv f \in \mathcal{O}_X(U)$ $\qquad \underline{\deg(f)=d} \longrightarrow$ multiplicity at 0



Find roots of $f(z_1, \underline{\underline{z_2}}) : b_1(z_1) \cdots b_d(z_1)$

$\sigma_k(z_1) = \sum_{i=1}^{d} b_i^k(z_1) \quad (k \geq 0)$ $\qquad \sigma_k = \frac{1}{2\pi i} \int_{|z_2|=\varepsilon} \frac{\xi_2^k \, \partial_{\xi_2} f(z_1, \xi_2)}{f(z_1, \xi_2)} d\xi_2$

$\downarrow$

holomorphic

$$\Longrightarrow \underset{\mathcal{O}_X(U)}{\overset{}{h}} = \left(z_2 - b_1(z_1)\right) \cdots \left(z_2 - b_d(z_1)\right)$$

$$= z_2^d - \underbrace{(b_1 + \cdots + b_d)}_{\sigma} z_2^{d-1} + \cdots + \underbrace{b_1 \cdots b_d}_{\sigma_d}$$

$f = u\underset{\substack{\downarrow \\ \text{Weierstrass polynomial}}}{h}$ $\qquad 0 \neq u$ is a unit in $\mathcal{O}_X(U)$

Given $f \not\equiv 0$ as above $\quad f = uh$

for $\forall g \in \mathcal{O}_X(U)$. $g = v \cdot h + r \Rightarrow \deg(r) < d$. $r \in \mathcal{O}_{n-1}[z_2]$

$\deg(h) = d$

To show $\mathcal{O}_n$ is Noetherian by dimension induction

$n=0$ ✓ $\qquad$ Assume holds for $n-1$, For $0 \neq \underset{f(x)=0}{\underline{f}} \in \mathcal{O}_n$ $\quad f = uh \quad d \geq 1$

$\forall g \in \mathcal{O}_n \quad g = v \cdot h + r$, $r \in \mathcal{O}_{n-1}[z_n]$

$\Rightarrow \forall g \in \mathcal{O}_n \Rightarrow g \in \mathcal{O}_{n-1}[h, z_n] \longrightarrow$ Noetherian

$\mathcal{O}_{n-1} \downarrow$ Noetherian

$(X, \mathcal{O}_X) \longrightarrow$ structure sheaf

$X \downarrow$ topological space

F is a sheaf of $\underline{\text{Abelian group}}$ on X

↓

has enough injective objects ⟶

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "E"] \\ & I & \end{tikzcd}$$

of Abelian groups

⟹ ∃ an injective resolution

$$0 \longrightarrow F \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

$\underline{\underline{\text{exact}}}$ seq of sheaves, I^i injective.

↕

$$(0 \to F_x \to I^0_x \to I^1_x \to \cdots)$$

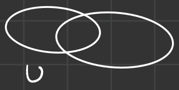
global section functor: $\underline{\Gamma}$

$$0 \longrightarrow \Gamma(I^0) \xrightarrow{d_0} \Gamma(I^1) \xrightarrow{d_1} \Gamma(I^2) \xrightarrow{d_2} \cdots \qquad d_i \circ d_{i-1} = 0, \quad i \geq 1$$

1. sheaf cohomology: $H^i(X, \underset{\mathbb{R},\,\mathbb{C}}{\boxed{F}}) = \mathrm{Ker}(d_i) / \mathrm{Im}(d_{i-1})$

acyclic

In particular, $F = \mathbb{R}$. locally constant sheaf with coefficient $\mathbb{R}$



$$0 \longrightarrow \mathbb{R} \longrightarrow \underset{\substack{\downarrow \\ \text{smooth functions}}}{A^0_X} \xrightarrow{d} A^1_X \xrightarrow{d} A^2_X \xrightarrow{d} \cdots \xrightarrow{d} \boxed{A^m_X} \to 0.$$

$m = \dim_{\mathbb{R}} X$

← de Rham complex

$$0 \to \Gamma(A^0_X) \xrightarrow{d} \Gamma(A^1_X) \xrightarrow{d} \cdots \Gamma(A^m_X) \longrightarrow 0$$

2. $\underline{\text{de Rham cohomology}}$ $H^i_{dR}(X, \mathbb{R}) = \mathrm{Ker}(d_i) / \mathrm{Im}(d_{i-1}).$

(Topological information)

X complex manifold $\quad d = \underset{(1,0)}{\partial} + \underset{(0,1)}{\bar{\partial}}$

Let $A^{p,q}_X$. sheaf of smooth (p,q) forms. $\qquad (A^k_X)_{\mathbb{C}} \simeq \bigoplus_{p+q=k} A^{p,q}_X$

Ω^p_X. sheaf of holomorphic p-forms

$$0 \longrightarrow \Omega^p_X \longrightarrow A^{p,0}_X \xrightarrow{\bar{\partial}} A^{p,1}_X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} A^{p,n}_X \xrightarrow{\bar{\partial}} 0$$

$n = \dim_{\mathbb{C}}$

← Dolbeault complex

$$0 \longrightarrow \Gamma(A_X^{p,0}) \xrightarrow{\bar\partial_0} \Gamma(A_X^{p,1}) \xrightarrow{\bar\partial_1} \cdots \longrightarrow \Gamma(A_X^{p,n}) \xrightarrow{\bar\partial_n} 0$$

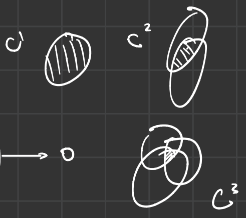
3. Dolbeault cohomology $H^i(X,\Omega^p) = \ker(\bar\partial_i)/\mathrm{Im}(\bar\partial_{i-1})$

(analytic information)

finite cover

Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open cover of X, s.t. $\phi \neq U_{\alpha_0}\cap\cdots\cap U_{\alpha_p} \simeq \mathbb{R}^m$. $m=\dim_{\mathbb{R}}(X)$.

Sheaf $\mathcal{F}=\mathbb{R}$, A_X^k. $C^p(\mathcal{U},\mathbb{R}) = \prod_{\substack{\alpha_0\cdots\alpha_p\\ \in I}} \mathbb{R}(U_{\alpha_0,\cdots\alpha_p})$



Čech complex: $0\longrightarrow C^0(\mathcal U,\mathbb R)\xrightarrow{\delta} C^1(\mathcal U,\mathbb R)\xrightarrow{\delta}\cdots\xrightarrow{\delta} C^k(\mathcal U,\mathbb R)\longrightarrow 0$

$\sigma\in\mathbb R(U_{\alpha_0\cdots\alpha_p})$, $\delta(\sigma)_{\alpha_0,\cdots\alpha_{p+1}} = \sum_{0\le i\le p+1}(-1)^i \sigma_{\alpha_0\cdots\hat\alpha_i\cdots\alpha_{p+1}}$

4. Čech cohomology: $H^i_{\check{C}h}(X,\mathbb R) = \ker(\delta_i)/\mathrm{Im}(\delta_{i-1})$

(combinatorial information) $\ulcorner H^i_{\check{C}h}(X,\mathbb C)\lrcorner$

Sheaf cohomology: $H^i(X,F)$ /R, C

Acyclic resolution of F. $0\to F\to \mathcal A^0 \xrightarrow{d_0}\mathcal A^1\xrightarrow{d_1}\mathcal A^2\xrightarrow{d_2}\mathcal A^3\xrightarrow{d_3}\cdots$

exact sequence, $H^i(X,\mathcal A^i)=0$ for $\forall i>0$

Lemma: A sheaf is called a fine sheaf if it admits a partition of unity

$s_\alpha\in A_X^k(U_\alpha)$ partition of unity $\rho_\alpha\in C^\infty(X)$ $\sum\rho_\alpha=1$

supp $\rho_\alpha\subseteq U_\alpha$

$s=\sum_\alpha \rho_\alpha s_\alpha$

$\forall A_X^k$ $(k\geq 0)$ is a fine sheaf

$\mathbb R$ is not fine.

Lemma For an acyclic resolution of F.

$$0 \to \Gamma(\mathcal{A}^0) \xrightarrow{d_0} \Gamma(\mathcal{A}^1) \xrightarrow{d_1} \cdots \Gamma(\mathcal{A}^k) \longrightarrow \cdots$$

induces $\Longrightarrow H^i(X, \mathcal{A})$

Cor:

$$H^i_{dR}(X.\mathbb{R}) \simeq \underset{\substack{\downarrow \\ \text{sheaf cohomology}}}{H^i(X.\mathbb{R})}$$

$$H^i(X, \mathcal{A}) \simeq H^i(X.F) \quad \text{for } i \geq 0$$

hypercohomology : $C^k = \bigoplus_{p+q=k} C^{p.q}$

$$D = (-1)^p \delta + d \Rightarrow D \circ D = 0$$

$$\longrightarrow C^k \xrightarrow{D^2} C^{k+1} \cdots$$

$$H^i(C^\circ . D) \simeq \mathrm{Ker}(D^i) / \mathrm{Im}(D^{i-1})$$

$$H^i(F.d) \simeq H^i(C^\circ, D) \simeq H^i(F, \delta)$$

Sketch of proof

$$\begin{tikzcd}
 & & 0 \arrow[d] & & C^{0.0} \arrow[ddll] & \\
0 \arrow[r] & \Gamma(F) \arrow[r, "\delta"] \arrow[d, "d"'] \arrow[dr] & \Gamma(\mathcal{A}^0) \arrow[r, "\delta"] \arrow[d] \arrow[dr] & \Gamma(\mathcal{A}^1) \arrow[r, "\delta"] \arrow[d] \arrow[dr] & \Gamma(\mathcal{A}^2) \arrow[r, "\delta"] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma(I^0) \arrow[r] \arrow[d, "d"'] \arrow[dr] \arrow[ur, dashed] & \Gamma(I^{0.0}) \arrow[r] \arrow[d] \arrow[dr] & \Gamma(I^{0.1}) \arrow[r] \arrow[d] \arrow[dr] & \Gamma(I^{0,2}) \arrow[d] & C^{1.1} \arrow[dll, bend left] \\
0 \arrow[r] & \Gamma(I^1) \arrow[r] \arrow[d, "d"'] \arrow[dr] & \Gamma(I^{1.0}) \arrow[r] \arrow[d] \arrow[dr] & \Gamma(I^{1.1}) \arrow[r] \arrow[d] \arrow[dr] & \Gamma(I^{1.2}) \arrow[d] & \\
0 \arrow[r] & \Gamma(I^2) \arrow[r] & \Gamma(I^{2.0}) \arrow[r] & \Gamma(I^{2.1}) \arrow[r] & \Gamma(I^{0.2}) &
\end{tikzcd}$$

By using the double complex with a similar idea, we can also show that

$$\underset{\text{sheaf}}{\underline{H^i(X,\mathbb{R})}} \simeq H^i_{dR}(X,\mathbb{R}) \simeq H^i_{\check{C}h}(X,\mathbb{R})$$

$$\boxed{\begin{array}{ll} H^k_{dR}(X.\mathbb{R}) \ni [\sigma] & \sigma \in A^k(X) \\ \downarrow & d\sigma = 0 \\ A^k(X) \ni \text{“}\sigma\text{”} & \sigma + \textcircled{ds} \end{array}}$$

$$H^i_{dR}(X.\mathbb{C}) \simeq H^i_{dR}(X.\mathbb{R}) \otimes \mathbb{C}$$

$$H^i_{DoR}(X. \Omega^P).$$

$$\begin{tikzcd}
T\left((A^k_X)_{\mathbb{C}}\right) \arrow[d, "d = \partial + \bar\partial"] \arrow[r, phantom, "\simeq"] & \bigoplus_{p+q=k} A^{p.q}_X \arrow[d, "\bar\partial"] \\
(A^{k+1}_X)_{\mathbb{C}} & A^{p.q+1}_X
\end{tikzcd}$$

(locally)

$$\Omega^P \longrightarrow A^{p.0}_X \xrightarrow{\bar\partial} A^{p.1}_X \xrightarrow{\bar\partial} \cdots$$

clue: guess :

Hodge decomposition for compact Kalhen manifold.

$$H^i_{dR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X. \Omega^P). \quad \text{(globally)}$$

$\downarrow$

NOT true for general complex manifolds

反例　Hopf surface $\simeq S^1 \times S^3$

$$H^1_{dR}(X, \mathbb{C}) = \mathbb{C} \quad \overset{?}{\simeq} \quad \underline{H^1(X, \Omega^0_X)} \oplus \underline{H^0(X, \Omega^1_X)}$$

odd.　　　一样.　∴ even.

Hodge decomposition :

① Harmonic forms　X : smooth compact manifold, $\dim_{\mathbb{R}} X = m$. g : Riemann metric on X

$\underline{\alpha, \beta} \in A^k(X)$. smooth k-form on X

$* : A^k \longrightarrow A^{m-k}$　which satisfies　$\underbrace{\alpha \wedge (*\beta)}_{m\text{-form}} = \langle \alpha, \beta \rangle_g \, vol_g$

$x \in X$. $dx_1, \cdots, dx_m$.　$(dx_1 \wedge \cdots dx_k) \wedge \underbrace{*(dx_1 \wedge \cdots \wedge dx_k)}_{\overset{\parallel}{dx_{k+1} \wedge \cdots \wedge dx_m}} = dx_1 \wedge \cdots \wedge dx_m$

$\underset{\overset{\parallel}{(-1)^{k(m-k)}}}{*^2} : A^k \to A^k$, $(*)^{-1} = (-1)^{k(m-k)} (*)$, define $\underset{A^k \to A^{k-1}}{d^* = (-1)^k *^{-1} d *}$

Lemma　$\alpha \in A^{k-1}$, $\beta \in A^k$

$$\Rightarrow \int_X \langle d\alpha, \beta \rangle_g \, vol_g = \int_X \langle \alpha, d^*\beta \rangle_g \, vol_g$$

$$\begin{tikzcd} A^{k-1} \arrow[r, "d"] & A^k \arrow[l, bend left] \arrow[r, "d"] & A^{k+1} \arrow[l, bend left] \end{tikzcd}$$

$\mathrm{Im}(A^k \xrightarrow{d} A^{k+1})$

$A^k \not\simeq A^k / \mathrm{Im}(A^{k-1}) \oplus \mathrm{Im}(A^{k-1})$.

$$\begin{tikzcd} A^{k-1}_X \arrow[r, shift left, "d"] & A^k_X \arrow[l, shift left, "d^*"] \arrow[r, shift left, "d"] & A^{k+1}_X \arrow[l, shift left, "d^*"] \end{tikzcd}$$

$\Box_d = d^* d + d d^* : A^k_X \longrightarrow A^k_X$

$\forall \alpha, \beta \in A^k(X)$

$$\int_X \langle \Box_d \alpha, \beta \rangle_g \, vol_g = \int_X \langle d\alpha, d\beta \rangle_g + \langle d^*\alpha, d^*\beta \rangle_g \, vol_g = \int \langle \alpha, \Box_d \beta \rangle_g \, vol_g$$

Def　$H^k(X) = \{ \underset{\overset{\downarrow}{\text{harmonic } k\text{-form}}}{\alpha \in A^k(X)}, \ \Box_d \alpha = 0 \} \subseteq A^k(X)$

__Lemma__: $A^k_X(x) \simeq d(A^{k-1}(x)) \oplus d^*(A^{k+1}(X)) \oplus \mathcal{H}^k(X)$ $\quad \forall\ 0 \le k \le n$

__Cor__ : For any $k \le n$ $\quad H^k_{dR}(X.\mathbb{R}) \simeq \underline{\mathcal{H}^k(X)}$

↘ finite dimensional. $\mathbb{R}$-vector space.

$\Box_d : A^k \longrightarrow A^k$

$x \in X$, normal coordinate

$$\Box_d \sim -\sum_{i=1}^{m} \partial_{x_i}^2 \sim \text{symbol } -\sum_{i=1}^{n} \xi_i^2$$ elliptic operator (∀ elliptic operator is $\underline{\text{Fredholm}}$)

i.e. Ker $(\Box_d)$ is finite. dim

Im $(\Box_d)$ is closed, codim is finite.

Complex case $\quad A^{p,q-1}(x) \xrightarrow{\bar\partial} A^{p,q}(x) \xrightarrow{\bar\partial} A^{p,q+1}(x)$

$\boxed{\text{Cplx Mnfd } X \quad \text{Hermitian metric } h}$ $\quad T^{1,0}_X$

Define $\quad * : A^{p,q} \longrightarrow A^{n-p,n-q}$ which satisfies

$\alpha, \beta \in A^{p,q}$ $\quad \alpha \wedge (*\bar\beta) = (\alpha.\beta)_h \,\mathrm{vol}_g \quad \longleftarrow \frac{1}{2}\mathrm{Re}(h) = g$

$(n.n)$.

$(*)^{-1} = (-1)^k *$, it's easy to see.

$$\int_X (\alpha.\beta)_h \,\mathrm{vol}_g = \int_X \alpha \wedge (*\bar\beta)$$

$$\begin{tikzcd} A^{p,q} \arrow[r, shift left, "\bar\partial"] & A^{p,q+1} \arrow[l, shift left, "\bar\partial^*"] \end{tikzcd}$$

$\bar\partial^* = -*\partial*$

$\partial^* = -*\bar\partial*$

$$\int_X (\bar\partial\alpha, \beta)_h \,\mathrm{vol}_g = \int_X (\alpha, \bar\partial^*\beta)_h \,\mathrm{vol}_g$$

$\Longrightarrow$ Define $\quad \underset{\bar\partial\bar\partial^* + \bar\partial^*\bar\partial}{\underset{\parallel}{\Box_{\bar\partial}}} : A^{p,q}(x) \longrightarrow A^{p,q}(x)$, $\quad \int (\Box_\partial \alpha.\beta)_h = \int (\alpha.\Box_\partial \beta)_h$

$\mathcal{H}^{p,q}(x) = \mathrm{Ker}(\Box_{\bar\partial})$ is finite dim $\mathbb{C}$-vector space.

↓

Harmonic $(p.q)$ form.

__Prop__: $\quad H^q(X.\Omega^p) \simeq \mathcal{H}^{p,q}(x)$

"wanted"

Hodge decomposition $\quad \mathcal{H}^k(x) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(x)$

↓

X is kahler

Let X be compact complex manifold

$H^k_{dR}(X, \mathbb{R}) \simeq \mathcal{H}^k \leftarrow$ Harmonic forms

$$(\bar\partial\alpha . \beta)_h = (\partial . \bar\partial^*\beta)_h$$

$$\parallel$$

$$\bar\partial\alpha \wedge *\bar\beta = \alpha \wedge \bar\partial * \bar\beta = (\alpha, -*\partial*\beta)$$

$$\underline{H^q_{Dol}(X, \Omega^p) \simeq \mathcal{H}^{p,q}(X)}$$

$$\Downarrow$$

$$S$$

$\bar\partial^* = -*\partial*$

$$(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \Box_{\bar\partial} S = 0$$

$$-\sqrt{-1}\left(\bar\partial[\Lambda, \partial] + [\Lambda, \partial]\bar\partial\right)$$

$\partial^* = -*\bar\partial*$

<u>Duality</u> real smooth cpt.

Poincare duality $\to$ (1) $H^k_{dR}(X, \mathbb{R}) \simeq H^{m-k}_{dR}(X, \mathbb{R})^*$ where $m = \dim_{\mathbb{R}} X$

Sarre duality $\to$ (2) $H^q_{Dol}(X, \Omega^p) \simeq H^{n-q}_{Dol}(X, \Omega^{n-p})^*$ $n = \dim_{\mathbb{C}} X$

<u>pf of (2)</u>. $H^q_{Dol}(X. \Omega^p) \simeq \mathcal{H}^{p.q}(X)$

$A^{n-q}_{Dol}(X, \Omega^{n-p}) \simeq \mathcal{H}^{n-p, n-q}(X)$

Define $\bar{*} : \mathcal{H}^{p.q}(x) \to \mathcal{H}^{n-p. n-q}(x)$

$S \longrightarrow \overline{*S}$

$$\Box_{\bar\partial} * \bar\alpha = * \Box_{\bar\partial} \bar\alpha = * \overline{\Box_{\bar\partial}\alpha} = 0$$

Hodge decomposition

$$0 \le k \le 2n \qquad H^k(X)_{\mathbb{C}} \simeq \bigoplus_{p+q=k} \mathcal{H}^{p.q}(X)$$

Let X be a smooth compact <u>Kahler</u> mnfd

<u>Kahler</u> form : $\omega(\ ,\ ) = g(J\ ,\ )$

$\Updownarrow$

$d\omega = 0 \Rightarrow$ closed 2-form

nondegenerate $\omega(\xi, J\xi) \neq 0$ iff $\xi \neq 0$

locally $x \in U$. $\omega = \frac{\sqrt{-1}}{2} \sum_i (dz_i \wedge d\bar{z}_i) + O(r^2)$

(1.1)-form

$[\omega] \in H^1(X, \Omega^1_X)$

$\in H^2(X, \mathbb{C})$

$\xrightarrow[\text{computation}]{\text{direct}}$ $d^*\omega = 0$

$\bar\partial^*\omega = 0$

$\partial^*\omega = 0$

$\Rightarrow \omega$ is harmonic (1.1)-form

Def: We define the operators:

(1) $L: A^{p,q}(X) \longrightarrow A^{p+1,q+1}(X)$

$$\alpha \longrightarrow \omega \wedge \alpha$$

Remark: $L: \mathcal{H}^{p,q}(X) \longrightarrow \mathcal{H}^{p+1,q+1}(X)$

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + \omega \wedge d\alpha = 0.$$

(2) $\Lambda: A^{p,q}(X) \longrightarrow A^{p-1,q-1}(X)$

for $\alpha \in A^{p,q}(X)$. $\beta \in A^{p+1,q-1}(X)$

$$(\Lambda\alpha, \beta)_h = (\alpha. L\beta)_h$$

Prop (Hodge identity)

(1) $[\Lambda. \bar\partial] = -\sqrt{-1}\,\partial^*$ $\Rightarrow$ (3) $[L, \bar\partial^*] = -\sqrt{-1}\,\partial$

(2) $[\Lambda. \partial] = \sqrt{-1}\,\bar\partial^*$ $\Rightarrow$ (4) $[L, \partial^*] = \sqrt{-1}\,\bar\partial$

Pf: The statement is of local nature, we can pick $x \in X$, and pick a holomorphic coordinate $x=0 \in U \subseteq \mathbb{C}^n$. In particular, we can choose the coordinate s.t

$$\omega = \sum_i \frac{\sqrt{-1}}{2} dz_i \wedge d\bar z_i + O(r^2).$$

At $x=0$, $\alpha \in \Omega^{p.q}$, $L\alpha = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar z_i \wedge \alpha$

$$\Lambda(\alpha) = 2\sqrt{-1}\sum_i \underset{\displaystyle L_i}{\underset{\|}{L(\partial_i)}} \wedge \underset{\displaystyle L_{\bar i}}{\underset{\|}{L(\partial_{\bar i})}} \wedge \alpha$$

By duality (1) $\Rightarrow$ (3)

(2) $\Rightarrow$ (4)

$$(\underline{[\Lambda, \partial]}\alpha. \beta)_h = -\sqrt{-1}(\bar\partial^*\alpha. \beta)_h$$

$$\Lambda\partial\alpha - \partial\Lambda\alpha \qquad\qquad \| \;\; (\alpha, \partial\beta)_h$$

$$\| $$

$$(\partial\alpha, L\beta)_h - (\Lambda\alpha, \partial^*\beta)_h$$

$$= (\alpha. \partial^*\underline{L\beta - L\partial^*\beta})h = (\alpha. \sqrt{-1}\,\partial\beta)_h$$

$$\underset{\displaystyle [\partial^*, L]\beta}{\|}$$

By conjugation (1) $\Rightarrow$ (2)

$$\overline{[\Lambda, \bar\partial]} = [\Lambda, \partial]$$

$$\overline{-\sqrt{-1}\,\partial^*} = \sqrt{-1}\,\bar\partial^*$$

∴ 只需证 (1)

Reduce to prove (1)

$$\bar\partial f = \sum_{1\le i\le n} \frac{\partial f}{\partial \bar z_i} \otimes d\bar z_i$$

$\bar\partial \longrightarrow$ $\underline{\text{Symbol}}$ $\sum_{1\le i\le n} d\overline{z_i}$

reduced to $[\Lambda, \sum_{1\le i\le n} d\bar z_i] = -\sqrt{-1}\,\partial^* = \sqrt{-1} * \bar\partial *$

$= \sqrt{-1} * \left(\sum_{1\le i\le n} d\overline{z_i}\right) *$

reduced to $[\Lambda, d\bar z_1] = \sqrt{-1} * (d\bar z_1) *$

$$*^{-1}[\Lambda, d\bar z_1] *^{-1} = \sqrt{-1}\, d\bar z_1$$

$$\Updownarrow$$

$$*[\Lambda, d\bar z_1]* = * \left(*^{-1} L * \circ d z_1 - d z_1 \circ *^{-1} L *\right) *$$

$$\overset{\|}{*^{-1}L*} \qquad = L * \circ d\bar z_1 - * d\bar z_1 \circ \boxed{*^{-1}} L \underline{*^2} \qquad \overset{* \cdot \underline{(*)^{-2}}}{}$$

$$= L \underset{\overset{\|}{2\iota(\partial_{z_1})}}{\underline{* d\overline{z_1} *}} - \underset{\overset{\|}{2\iota(\partial_{z_1})}}{\underline{* d\bar z_1 *}} L$$

α contains dz_1 ?

???

$\alpha = dz_1$

$* d\bar z_1 \wedge * \alpha = d\bar z_1 \wedge (-1) dz_1 \wedge$

$\cdots \wedge dz_n \wedge d\bar z_2$

$\vdots \qquad \cdots \wedge d\bar z_n$

$\underline{\text{Cor}}$: $\square_d = d^* d + d d^*$ $\qquad \mathrm{Ker}(\square_d) \simeq \bigoplus_k \mathcal{H}^k$

$\square_{\bar\partial} = \bar\partial^* \bar\partial + \bar\partial \bar\partial^*$ $\qquad \mathrm{Ker}(\square_{\bar\partial}) \simeq \bigoplus_{p,q} \mathcal{H}^{p,q}$

$\square_\partial = \partial^* \partial + \partial \partial^*$

(Hodge identity) $\Rightarrow$ $\square_d = 2\square_{\bar\partial} = 2\square_\partial$

$$\square_{\bar\partial} = -\sqrt{-1}\left([\Lambda, \partial]\bar\partial + \bar\partial[\Lambda, \partial]\right)$$

$$= -\sqrt{-1}\left(\Lambda \underset{\overset{\|}{-\bar\partial\partial}}{\boxed{\partial\bar\partial}} - \partial\Lambda\bar\partial + \bar\partial\Lambda\partial - \underset{-\partial\bar\partial}{\boxed{\bar\partial\partial}}\Lambda\right)$$

$$= -\sqrt{-1}\left(\underset{\overset{\|}{-[\Lambda, \bar\partial]}}{(-\Lambda\bar\partial + \bar\partial\Lambda)}\partial + \partial\underset{\overset{\|}{-[\Lambda, \bar\partial]}}{(-\Lambda\bar\partial + \bar\partial\Lambda)}\right)$$

$\alpha *$

$d = \partial + \bar\partial$

$$\square_d = \underset{\overset{\|}{\partial^* + \bar\partial^*}}{(\partial + \bar\partial)^*}(\partial + \bar\partial) + (\partial + \bar\partial)(\partial + \bar\partial)^*$$

$$= \partial^*\partial + \bar\partial^*\bar\partial + \partial\partial^* + \bar\partial\bar\partial^*$$

$$\underset{\overset{\|}{-\bar\partial\partial^*}}{\partial^*\bar\partial} + \bar\partial\partial^* = 0$$

Cor: We have Hodge decomposition $\quad H^k_{\mathbb{C}} \simeq \bigoplus_{p+q=k} H^{p,q}(X)$

$\underline{\text{Pf}}$: $\mathcal{H}^k_{\mathbb{C}} \xrightarrow[\text{injective}]{\text{surjective}} \bigoplus_{p+q=k} A^{p,q}(x)$ $\qquad T^*X_{\mathbb{C}} \simeq \oplus (T^{1,0})^* \oplus (T^{0,1})^*$

$\rotatebox{90}{$\in$}$

$S = \sum_{p+q=k} S_{p,q}$, $0 = \square_d S = \sum \boxed{\square_d S_{p,q}} = 0$ $\Rightarrow \square_{\bar\partial} S_{p,q} = 0 \Rightarrow S_{p,q} \in \mathcal{H}^{p,q}$

$\forall\, S \in \mathcal{H}^{p,q} \quad \rightarrow \quad \square_d S \Rightarrow \square_{\bar\partial} S = 0 \quad \Rightarrow S \in \mathcal{H}^k_{\mathbb{C}}$

$\forall\, s \in H^b_{\mathbb{C}} \;,\; f(s) = \bigoplus_{p+q=k} S_{p,q} \;,\; f(s) = 0 \quad \Rightarrow S_{p,q} = 0.$

<u>Corollary</u> : $H^q_{Dol}(X, \Omega^p) \simeq H^p_{Dol}(X, \Omega^q)$

<u>Proof</u> :

$$\begin{tikzcd} H^q_{Dol}(X,\Omega^p) \arrow[d, phantom, sloped, "\simeq"] & H^p_{Dol}(X,\Omega^q) \arrow[d, phantom, sloped, "\simeq"] \\ \mathcal{H}^{p,q} \arrow[r, phantom, "\simeq"] & H^{q,p} \end{tikzcd}$$

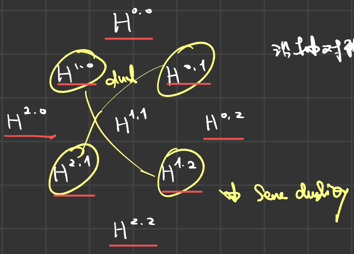
$$\mathcal{H}^{p,q} \xrightarrow{\;-\;} \mathcal{A}^{q,p}(X)$$

$$s \longrightarrow \bar{s}$$

$0 = \square_{\bar\partial} s = \square_d s \Rightarrow \overline{\square_d s} = \square_d \bar{s} = 0 \quad \bar{s} \in \mathcal{H}^{q,p}$ $\square$

e.g. $H^*(X, \Omega^*) = \bigoplus_{p,q} H^q(X, \Omega^p)$

$n = 2.$



沿轴对称 (一样).

Kahler $\Rightarrow$ $H^1(X, \mathbb{C})$ is of even dimension

$H^1(X, \mathbb{C})$ odd dim $\Rightarrow$ X is not Kahler

Lefschetz decomposition

$L : A^k_X \longrightarrow A^{k+2}_X$ Define $V_k = A^{k+n}_X$ $\quad -n \le k \le n$

$\Lambda : A^{k+2}_X \longrightarrow A^k_X$ $\quad V = \bigoplus_{-n \le k \le n} V_k$ $\quad L, \Lambda : V \longrightarrow V$

$[L, \Lambda](\alpha) = 2k\alpha$ $\quad H : V \longrightarrow V$

$\alpha \in V_k$ $\quad \alpha \in V_k \longrightarrow H(\alpha) = k\alpha$

$$\left\{\begin{array}{l} [L, \Lambda] = 2H \\ [H, L] = 2L \\ [H, \Lambda] = -2\Lambda \end{array}\right.$$

$$\begin{array}{ccc} L, & H, & \Lambda \\ / & \| & \backslash \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{array} \in \mathfrak{sl}(2, \mathbb{C})$$

$\Rightarrow$ <u>$\mathfrak{sl}(2\,\mathbb{C})$ representation on V.</u>

$\Downarrow$

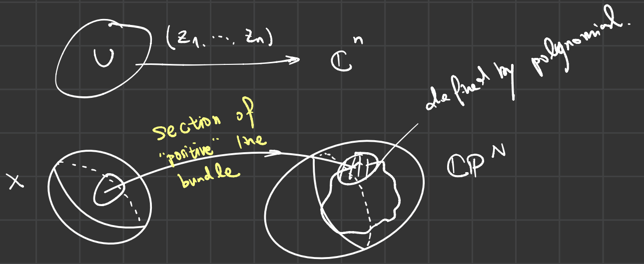
Lefschetz decomposition $\rightarrow$ Hodge index theorem

Kahler mnfd (X, ω) ← positive (1,1)-form

"Quantize a kahler manifold by "positive line bundle" "

$$f = \lim_{N\to\infty} \sum_{k=0}^{N} a_k z^k \quad \Big| \quad f = \sum a_k x^k$$

analytic fn.



X : Kahler manifold

A line bundle $\mathbb{C}^1 \to L \downarrow X$ is a <u>holomorphic vector bundle</u> of rank 1.

$$\left(\begin{tikzcd} \mathbb{C}^r \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}, \ h \right) \overset{\exists!}{\rightsquigarrow} \text{Chern connection} \left((\nabla^c)^{0,1} = \bar\partial \right)$$

← Hermition metric

$$\nabla = \nabla^{1,0} + \nabla^{0,1}$$
$$= (\partial + h^{-1}\partial h) + \bar\partial$$

local holomorphic frame

$$F(h) = \nabla \circ \nabla = (d + h^{-1}\partial h)(d + h^{-1}\partial h)$$
$$= d(h^{-1}\partial h) + (h^{-1}\partial h) \wedge (h^{-1}\partial h)$$
$$\overset{\shortparallel}{\partial(h^{-1}\partial h) + \bar\partial(h^{-1}\partial h)}$$
$$\overset{\shortparallel}{-h^{-1}\partial h \wedge h^{-1}\partial h}$$

rank 1 case : L frame S_1

$$= \bar\partial(h^{-1}\partial h)$$

h is a function $\quad h(S_1, S_1) = h_{11} S_1 \bar{S}_1$

$$h^{-1}\partial h = \partial \log h$$

$$\boxed{F(h) = \bar\partial \partial \log h = -\partial\bar\partial \log h.}$$

In particular, anti-canonical line bundle : $-K_X \simeq \Lambda^n T^{1,0}X$ is line bundle

metric on $-K_X$ is determinant

$$h(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) = (\xi_1 \wedge \bar\xi_1) \wedge \cdots \wedge (\xi_n \wedge \bar\xi_n)$$

$$\eta = B\xi. \qquad \frac{h(\eta, \eta)}{h(\xi, \xi)} = |\det(B)|^2$$

Use local holomorphic normal frame $(S_1, \cdots, S_n)$ of $T^{1,0}X$. $\qquad \omega = (g_{i\bar j})_{1 \le i, j \le n}$

$$\begin{tikzcd}(X, \omega) \arrow[d, squiggly] & & \\ \text{metric on } T^{1,0}X \arrow[r, squiggly] & \text{metric } h \text{ on } -K_X\end{tikzcd}$$

$\xi = \sum \alpha_i s_i$ $\quad \alpha_1 \alpha_2 \cdots \alpha_n s_1 \wedge \cdots \wedge s_n$

$//$

$h(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n, \underbrace{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n)}\ ) \quad \alpha_1 \cdots \alpha_n \cdot \overline{\alpha_1 \cdots \alpha_n}$

$= \det(g_{i\bar{j}}) \qquad |\alpha_1|^2 \cdot |\alpha_2|^2 \cdots |\alpha_n|^2$

$F(h_{-K_X}) = -\partial\bar{\partial} \det(g_{i\bar{j}})$

$Ric(\omega) = \sqrt{-1}\, F(h_{-K_X}) = -\sqrt{-1}\, \partial\bar{\partial} \det(g_{i\bar{j}})$

$Ric(g) = 2 Ric(\omega)$

remark: Kahler-Einstein metric : $Ric(\omega) = \lambda \cdot \omega$

A line bundle L on X is called "positive line bundle"

if it admits a Hermitian a metric h on L s.t.

$\sqrt{-1}\, F(h)$ is a positive (1.1)-form.

If a complex mnfd X admits a positive line bundle

then X is Kahler.

Take $\omega = \sqrt{-1}\, F(h)$

$\Rightarrow$ ω is d-closed.

$H^1(X, \underline{\mathcal{O}_X^*}) = \{\text{line bundles}\}$

$\searrow$ nowhere vanished holomorphic functions

$\cong$

$H^0(X, \mathcal{M}^*/\mathcal{O}_X^*)$

$\searrow$ is $\{\text{divisors}\}$

positive line bundle $L^{\otimes k}$, $S_0, \dots, S_N \in H^0(X, L^{\otimes k})$

$$\left.\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^N \\ x & \longmapsto & [S_0(x), \dots, S_N(x)] \end{array}\right\} \Rightarrow \text{defined by polynomials.}$$

Line bundle
$$\begin{tikzcd} \mathbb{C}^1 \arrow[r] & L \arrow[d] \\ & X \end{tikzcd}$$
L — holo vector bundle of $rk(L) = 1$

X — complex, cpt. Mnfd.

$L \longleftrightarrow$ transition maps

$\{U_\alpha\}_{\alpha \in I}$ a cover of X

$U_\alpha \subseteq \mathbb{C}^n$

$U_\alpha \cap U_\beta : \pi^{-1}(U_\alpha) \xrightarrow{\Phi_{\alpha\beta}} \pi^{-1}(U_\beta)$

$0 \neq \Phi_{\alpha\beta}|_x \in \mathbb{C}$

$\Phi_{\alpha\beta}$: a holomorphic function on $U_\alpha \cap U_\beta$ and $\Phi_{\alpha\beta} \neq 0$;

$\{\Phi_{\alpha\beta}\}_{\alpha,\beta \in I}$

Denote $\mathcal{O}_X$: sheaf of holo func.

$\mathcal{O}_X^*$ nowhere vanished holo functions

i.e. $U \subset X$.

$\mathcal{O}_X^*(U) = \{ f \in \mathcal{O}_X(U) : f(x) \neq 0, \forall x \in U \}$

$\Longrightarrow$ from the views of transition maps.

line bundle $L \longleftrightarrow \{\Phi_{\alpha,\beta}\}_{\alpha,\beta \in I} \in \underline{H^1_{\check{C}h}(X, \mathcal{O}_X^*)}$

Prop: $\{$line bundles on $X\}$ $\xleftrightarrow{1-1}$ $H^1_{\check{C}h}(X, \mathcal{O}_X^*)$

$\downarrow$ group structure

("0" $= \mathcal{O}_X$, "+", "−")

isomorphism of Abelian group.

line bundles L, H

(1) $L^\vee = Hom(L, \mathcal{O}_X)$ denoted by $-L$; (or $L^{\otimes(-1)}$)

(2) $L \otimes H$, denoted as $L + H$

$\cong$

$H \otimes L$.

Consider the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp(2\pi i)} \mathcal{O}_X^* \to \overset{\text{or } o.}{0}$$

induces long exact sequence of cohomology

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^1(X,\mathbb{Z})$$

$$\begin{tikzcd}
{} \arrow[r] & H^1_{\check{C}h}(X,\mathcal{O}_X) \arrow[r] & H^1_{\check{C}h}(X,\mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[dr, "c_1(L)"'] & H^2_{\check{C}h}(X,\mathbb{Z}) \arrow[r, "i_*"] \arrow[d] & H^2_{\check{C}h}(X,\mathcal{O}_X) \arrow[d, "\wr S"] \\
& & & H^2_{dR}(X,\mathbb{R}) \arrow[r, "\pi^{0,2}"] & H^2_{Dol}(X,\mathcal{O}_X)
\end{tikzcd}$$

Čech cohomology

(1,1) form $F(h)$

projection to $(0,2)$

Recall (L,h) : $F(h) = -\partial\bar\partial \log h \in H^2_{dR}(X,\mathbb{R})$

$$\begin{tikzcd}
H^2_{\check{C}h}(X,\mathbb{R}) \arrow[r, phantom, "\simeq"] & H^2_{dR}(X,\mathbb{R}) \\
H^2_{\check{C}h}(X,\mathbb{Z}) \arrow[u] &
\end{tikzcd}$$

$$\begin{array}{l} H^2_{dR}(X,\mathbb{R})_{\mathbb{C}} \\ \rotatebox{90}{$\in$} \\ \sigma \quad \underline{2\text{-form}} \quad d\text{-closed} \\ (2,0) \oplus (1,1) \oplus (0,2) \end{array}$$

$$H^2_{Dol}(X,\mathcal{O}_X) : 0 \to A^0_X \xrightarrow{\bar\partial} A^{0,1}_X \xrightarrow{\bar\partial} A^{0,2}_{\cancel{X}} \xrightarrow{\bar\partial} \cdots$$

<u>notation</u> : $\begin{array}{c} L \\ \downarrow \\ X \end{array}$, $F(h) = -\partial\bar\partial \log h \in H^2(X,\mathbb{R})$

$$\boxed{[F(h)] =: c_1(L)}$$, 1st-Chern class.

Chern-Weil form (characteristic classes)

<u>Prop</u>: (1)

$$\begin{tikzcd}
H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1(L)"] \arrow[dr, "\delta"'] & H^2_{dR}(X,\mathbb{R}) \\
& H^2_{\check{C}h}(X,\mathbb{Z}) \arrow[u]
\end{tikzcd}$$

Proof. Le $H^1(X, \mathcal{O}_X^*)$

$\Updownarrow$

$\{\Phi_{\alpha\beta}\}_{\alpha.\beta\in I}$

constant on $U_\alpha\cap U_\beta\cap U_\gamma$

$$H^2(X,\mathbb{Z})\ni(\delta\Phi)_{\alpha\beta\gamma}=\boxed{\log\Phi_{\alpha\beta}+}\log\Phi_{\beta\gamma}+\log\Phi_{\gamma\alpha}$$

$$d(\delta\Phi)_{\alpha\beta\gamma}=d\log\Phi_{\alpha\beta}+d\log\Phi_{\beta\gamma}+d\log\Phi_{\gamma\alpha}=0$$

$\shortparallel$

$$\delta(d\log\Phi)_{\alpha\beta\gamma}=(d\log\Phi)_{\alpha\beta}+(d\log\Phi)_{\beta\gamma}+(d\log\Phi)_{\gamma\alpha}$$

$\Rightarrow$ $d\Phi$ is compatible on $U_{\alpha\beta\gamma}$

$\underline{U_{\alpha\beta}}$ $\{S_\alpha\}_{\alpha\in U}$ is orthonormal frame

?? $U_{\alpha\beta}\vdash$ $h=\Phi_{\alpha\beta}\ \underset{\underset{\Phi_{\alpha\beta}}{\shortparallel}}{\bar{\Phi}_{\alpha\beta}}$

$$(d\log\Phi)_{\alpha.\beta}=(\partial\log\Phi)_{\alpha\beta}=(\partial\log(\underset{\underset{h}{\shortparallel}}{\underline{\Phi.\bar\Phi}}))_{\alpha\beta}=(\partial\log h)_{\alpha\beta}=(\underset{A''}{\underline{h^{-1}\partial h}})_{\alpha\beta}$$

Chern connection

Recall: $\nabla^c(h)=\underset{A''}{\underline{h^{-1}\partial h}}+\underset{\underset{d}{\shortparallel}}{\underline{\partial+\bar\partial}}$

$$\underset{\underset{\partial+\bar\partial}{\shortparallel}}{\underline{d}}(h^{-1}\partial h)_\alpha=(F(h))_\alpha=(\bar\partial(h^{-1}\partial h)+h^{-1}\partial h\wedge h^{-1}\partial h)_\alpha$$

$$(\delta\underset{\underset{d(h^{-1}\partial h)}{\shortparallel}}{\underline{F(h)}})_{\alpha\beta}=d(\underset{\underset{d\log\Phi_{\alpha\beta}}{\shortparallel}}{\underline{\delta h^{-1}\partial h}})_{\alpha\beta}=\underset{\sim}{d^2}\log\Phi_{\alpha\beta}=0$$

$\therefore$ $(F(h))_\alpha$ is compatiable on $U_{\alpha\beta}$ $\longrightarrow$ $F(h)$ is global defined.

$\square$

Rmk: $\mathrm{Im}(\delta) \simeq \mathrm{Ker}(i_*)$

Neron - Sever. group

(2) $H^1(X, \mathcal{O}_X^*) / H^1(X, \mathcal{O}_X) \simeq \underline{H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})}$

$\swarrow$

$\{\text{line bundle}\} / \text{linear equivlence.}$

On the other hands: $\mathcal{M}_X$ = sheaf of meromorphic functions

$\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ = sheaf of $\underset{\text{non-vanished}}{\wedge}$ meromorphic functions.

$\Rightarrow$ exact seq

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^* / \mathcal{O}_X^* \longrightarrow 0$$

$\Rightarrow$ long exact seq

$$\cdots \longrightarrow H^0(X, \mathcal{M}_X^*) \longrightarrow \underline{H^0(X, \mathcal{M}_X^* / \mathcal{O}_X^*)} \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*)$$

f_α / g_α, f_α, g_α are holo function $(\in \mathcal{O}_X(U_\alpha))$

$\{h_\alpha\}_\alpha$

$h_\alpha \sim h_\alpha u_\alpha.$

$\underline{0 \neq u_\alpha \in \mathcal{O}_X^*(U_\alpha)}$

$\{\text{divisors}\}$

"除子"

$\downarrow$

$\{h_\alpha = 0\} - \{h_\alpha^{-1} = 0\}$

e.g. divisor. denoted by $D = \underbrace{\{h_\alpha = 0\}}_{\text{zero locus}}{}_{\alpha \in I}$

$\downarrow \delta$

<u>line bundle</u> $\mathcal{O}_X(D) \in H^1(X, \mathcal{O}_X^*)$

$\|$

$\{\Phi_{\alpha\beta}\}_{\alpha, \beta \in I}$

$\|$

$\dfrac{h_\beta}{h_\alpha} = \mathcal{O}^*(U_{\alpha\beta})$

Rmk: When X is projective, $H^0(X, \mathcal{M}_X^* / \mathcal{O}_X^*) / H^0(X, \mathcal{M}_X^*) \simeq H^1(X, \mathcal{O}_X^*)$

i.e. $\forall$ line bundle L on X

can be represented by a <u>divisors</u>

Recall: $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is "positive" if $\exists\, h$ s.t. $\sqrt{-1}F(h) > 0$

Thm Let X be a cpt Kahler mnfd, (L, h) is positive; then $\exists$ positive integer m, s.t. $\exists$ sections $S_0, \cdots, S_N \in H^0(X, mL)$ that gives an embedding:

$$mL = L^{\otimes m}$$

$$X \xrightarrow{f} \mathbb{P}^N$$

$$x \longrightarrow [S_0(x), \cdots, S_N(x)]$$

$$(S_0(x), \cdots, S_N(x)) \in \mathbb{C}^{N+1} \setminus \{0\}$$

idea of proof

If "f" is well-defined, then $\forall p \in X$, $\exists S_i \in H^0(X, mL)$ s.t. $S_i(p) \neq 0$

direct image

$$\left.\begin{matrix} \iota_X : p \longrightarrow X \\ \iota_{X_*} : \mathbb{C} \longrightarrow \mathcal{O}_X \end{matrix}\right\} \Rightarrow 0 \to m_p \to \mathcal{O}_X \to \mathcal{O}_X/m_p \to 0$$

$$\boxed{\iota_{p_*}(\mathbb{C}) \simeq \mathcal{O}_X/m_p}$$

skyscraper sheaf

denote: $m_p = \{ f \in \mathcal{O}_X ;\ p \in U,\ f \in \mathcal{O}_p(U),\ f(p) = 0 \}$

$$\begin{matrix} m_p \\ \cap \\ \mathcal{O}_X \end{matrix}$$

$$0 \to m_p \to \mathcal{O}_X \to \mathcal{O}_X/m_p \to 0$$

$$\otimes mL$$

$$0 \to mL \otimes m_p \to mL \to mL \otimes \mathcal{O}_X/m_p \to 0$$

surjective $\Leftrightarrow$ $S_i(p) \neq 0$

$$H^0(X, mL) \longrightarrow \underline{H^0(X, mL \otimes \mathcal{O}_X/m_p)} \longrightarrow \boxed{H^1(X, mL \otimes m_p)} = 0$$

$$\begin{matrix} s & & \begin{matrix} \wr\!\wr \\ H^2(p, \mathbb{C}) \simeq \mathbb{C} \\ S(p) \end{matrix} \end{matrix}$$

sheaf, not line bundle

To show that f is well-defined, it suffices $H^1(X, mL \otimes \underline{m_p}) = 0$

2 ways: $\left\{ \begin{array}{l} \text{(1) vanishing theorem of multiplier ideal sheaf} \\ \text{(2) Blow up construction:} \end{array} \right.$

transform $mL \otimes m_p \longrightarrow$ line bundle ?

$X \xrightarrow{f} \mathbb{P}^N$ well-defined

f is embedding $\left\{ \begin{array}{l} \text{(1) } f \text{ in injective} \\ \text{(2) } f_* \text{ is injective on } T_pX \text{ for any } p \in X \end{array} \right.$

(1) f injective, $\forall p \neq q \in X$, $\exists S \in H^0(X, mL)$ st $S(p) \neq S(q)$

$m_p \oplus m_q$ $\quad 0 \longrightarrow m_p \otimes m_q \longrightarrow \mathcal{O}_X \longrightarrow \underbrace{\mathcal{O}_X / m_p \otimes m_q}_{\cong\ \mathbb{C} \oplus \mathbb{C}} \longrightarrow 0$

$$H^0(X, mL) \xrightarrow{\text{surjective}} H^0(X, mL \otimes \mathcal{O}_X / m_p \otimes m_q)$$

$$\longrightarrow \underbrace{H^1(X, mL \otimes m_p \otimes m_q)}_{= 0,}$$

$\forall p \in X$, (cotangent bundle at p) $\simeq m_p / m_p^2$

$0 \longrightarrow m_p^2 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X / m_p^2 \longrightarrow 0$ $\quad \overset{\cong}{\text{1st order terms}}$

$dz_1, dz_2, \cdots, dz_n$

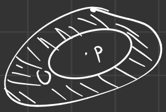
$$H^0(X, mL) \xrightarrow{\text{surjective.}} H^0(X, mL \otimes \mathcal{O}_X / m_p^2) \longrightarrow \underset{\overset{\|}{0}}{H^1(X, mL \otimes m_p^2)}$$

(1) "Nadel" $\sim$

(2) Kodaira Vanishing theorem ②

① Blow up construction :

$p \in X$. $Y = Bl_p X$



$$\begin{tikzcd} Y \arrow[d, "\pi."'] \\ X \end{tikzcd}$$

choose $p \in U \subseteq \mathbb{C}^n$

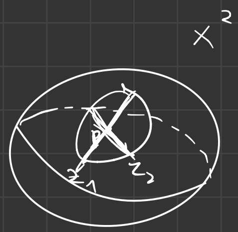
$$\tilde{U} = \{ ((z_1, z_2, \cdots, z_n), \underline{[u_1, u_2, \cdots, u_n]}) \in \mathbb{C}^n \times P^{n-1} \}$$

$z_i u_j = z_j u_i \quad \forall i,j$

$(0, \cdots, 0) \neq (u_1, u_2, \cdots, u_n) \sim (\lambda u_1, \cdots, \lambda u_n)$

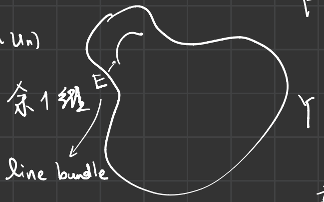
$\forall\, 0 \neq \lambda \in \mathbb{C}$

X^2

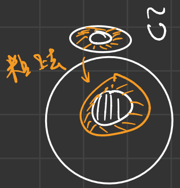


blow up

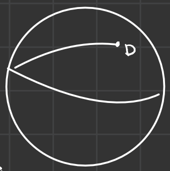
$P = \{ z_1 = 0 \} \cap \{ z_2 = 0 \}$.



$Y = \tilde{U} \cup (X \setminus U_{1/2})$



高余维



$E := \{ (0, \cdots, 0), [u_1, \cdots u_n] \in \tilde{U} \} \subseteq \tilde{U}$

$D = \{ h_\alpha = 0 \}$

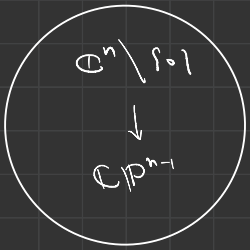
↓

line bundle

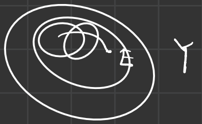
$$\begin{tikzcd} Y \arrow[d, "\pi:"'] \\ X \end{tikzcd}$$

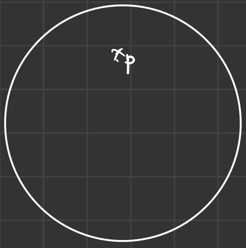
holomorphic → $\pi : Y \setminus E \xrightarrow{\text{biholomorphic}} X \setminus \{p\}$

E_δ open nbhd of E



total space of $\mathcal{O}(-1)$ $\mathbb{C}P^{n-1}$





$$\begin{tikzcd} \pi^*(mL) \arrow[r] \arrow[d] & mL \arrow[d] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

$$H^0(X, mL) = H^0(Y, \pi^*(mL))$$

(positive) line bundle.

$$\begin{tikzcd} H^0(Y, \pi^*(mL) \otimes \mathcal{O}_Y(-E)) \arrow[r] \arrow[d, phantom, "\cong"] & H^0(Y, \pi^*(mL)) \arrow[r, two heads, "\text{surjective}"'] \arrow[d, phantom, "\cong"] & H^0(E, \pi^* mL|_E) \arrow[r] \arrow[d, phantom, "\cong"] & \underline{H^1(Y, \pi^* mL - E)} \arrow[d, phantom, "\Vert"] \\ H^0(X, mL \otimes m_p) \arrow[r] & H^0(X, mL) \arrow[r, two heads, "!"] & H^0(X, mL \otimes \mathcal{O}_X / m_p) & 0 \end{tikzcd}$$

$\downarrow$

$H^0(Y, \pi^*(mL)-E)$

line bundle

$$\pi^*(m_p) \simeq \{ f \in \mathcal{O}_Y(U) : f|_{E\cap U} = 0 \}.$$

$$\simeq \mathcal{O}_Y(-E) \leftarrow \text{line bundle.}$$

(Kodaira Vanishing Thm) Let X cpt Kahler mnfd

② Let L positive line bundle. Then: $\underline{H^q(X, K_X \otimes L) = 0}$ for $\forall q \geq 1$.

$$\Downarrow$$

$$H^1(X, K_X \otimes L) = 0$$

Blow up

$$\begin{tikzcd} Y \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$$K_Y = \pi^* K_X + (n-1) E$$

$$\pi^*(mL) - E - K_Y + K_Y$$

①

$$= \underline{\left(\pi^*(mL - K_X) - nE\right)} + K_Y$$

$\swarrow$

positive when m is very large.

Kodaira embedding: cpt. Kahler mnfd X, positive line bundle L. $\exists$ positive integer m s.t. $S_0, S_1, \cdots, S_N \in H^0(X, mL)$. $X \xrightarrow{[S_0, S_1, \cdots S_N]} \mathbb{P}^N$

(1) $\underline{\pi^*(mL - K_X)} - \boxed{nE}$ — nW_E

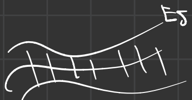
is a positive line bundle for m sufficiently large.

$\Vert$ $\pi^*(W_m) > 0$

$\pi^* W_m + n W_E$

$$\begin{tikzcd} & \mathbb{CP}^{n-1} \arrow[d, phantom, "\simeq"] & & \\ & E \subseteq Y = Bl_p X \arrow[d] & \arrow[d, "\pi"] & \\ & p \in X & & \end{tikzcd}$$

$$\mathbb{C}^n \setminus \{0\} \simeq \left(\begin{tikzcd} E_\delta \arrow[d] & \delta \text{ nbhd of } E \text{ in } Y \arrow[l] \\ \mathcal{O}(-1) \\ \mathbb{CP}^{n-1} \end{tikzcd} \right) \xrightarrow[\rho]{} \mathbb{CP}^{n-1}$$



$H^1(Y, \pi^*(mL) - E) = 0$ $\nearrow$ positive

$K_Y + \underbrace{\pi^*(mL) - E - K_Y}$ $\quad K_Y = \pi^* K_X + (n-1) E$

$\Longrightarrow H^1(Y, \underline{\pi^*(mL - K_X) - nE})$

"positive" $m \gg 1$

(2) Kodaira vanishing theorem: Let X be cpt Kahler mnfd. L is positive line bundle, then

$$H^q(X, K_X + L) = 0 \quad \text{for } \forall q > 0.$$
$$\parallel$$
$$H^q(X, \Omega^n \otimes L)$$

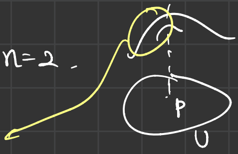
Adjunction formula:

$$\begin{tikzcd} Y = Bl_p X \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$$K_Y = \pi^* K_X + (n-1) E$$

(Exercise)

$n = 2$.



(z_2, v)

$\subseteq (z_1, z_2, \underbrace{[u_1, u_2]})$ $\quad u_1 z_2 = u_2 z_1$

(z_1, z_2)

$[\underbrace{\frac{u_1}{u_2}}, 1]\ (u_2 \neq 0)$ $\quad \Downarrow$

$\parallel$

v $\quad z_1 = z_2 \cdot v$

K_Y local section

$dz_2 \wedge dv$

K_X: local section

$dz_1 \wedge dz_2$

$$\pi^* K_X : \pi^*(dz_1 \wedge dz_2) = z_2 \underbrace{dv \wedge dz_2}_{K_Y} \Longrightarrow (2-1)E + \pi^* K_X = K_Y$$

$\parallel$

$v z_2$

$\{z_2 = 0\} \Longleftrightarrow -E$

$L \to \sqrt{-1} F(h_L) > 0$. on X

$K_X \to \sqrt{-1} F(h_{K_X})$: smooth (1.1) form

$\Big\} \overset{m \gg 1}{\Longrightarrow}$ $m\sqrt{-1} F(h_L) - \sqrt{-1} F(h_{K_X}) > 0$ on X

$\pi^*(\downarrow) > 0$ on $Y \setminus E$

$\pi^*(\quad) \geq 0$ on Y

$$\begin{tikzcd} \mathbb{CP}^{n-1} & \mathcal{O}(-1) \arrow[l, "\rho"'] \end{tikzcd} \qquad \rho^* \omega_{FS} \geq 0.$$

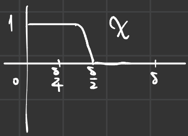
$$\begin{tikzcd} E_\delta \arrow[r, "\gamma"] & \mathcal{O}(-1) \end{tikzcd}$$

$$\omega_{FS} = \sqrt{-1}\, \partial\bar\partial \log\left(|\frac{z_0}{z_i}|^2 + \cdots + |\frac{z_{n-1}}{z_i}|^2\right) > 0$$

$\cap$

$\sqrt{-1}\, c_1(\mathcal{O}(1))$

$$\chi = \begin{cases} 1 & \rho(E_{\delta/4}) \\ 0 & \rho(E_{\delta/2}) \end{cases}$$



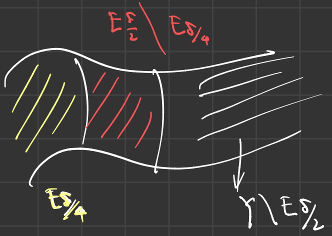
$$\omega_E = \gamma^*\left(\sqrt{-1}\, \partial\bar\partial \left(\chi \log\left(|\frac{z_0}{z_i}|^2 + \cdots + |\frac{z_{n-1}}{z_i}|^2\right)\right)\right)$$

$\downarrow$

(1-1)-form on Y $\qquad \sqrt{-1}\, c_1(E) \ni \omega_E \geq 0$ on $E_{\delta/4}$, $\omega_E|_E > 0$

$\pi^* \omega_m + n\,\omega_E$

$$\begin{cases} E_{\delta/4} & \omega_Y > 0 \\ Y \setminus E_{\delta/2} & \omega_Y > 0 \\ E_{\delta/2} \setminus E_{\delta/4} & \omega_Y > 0 \quad \text{when } m >> 1 \end{cases}$$



Key Lemma (Bochner – Weitzenbock formula): Let (X, ω) a Kahler mnfd

$\mathbb{C}^r \longrightarrow E$ a holomorphic vector bundle
$\quad\quad\quad\ \ \downarrow$
$\quad\quad\quad\ \ X$

$$\nabla_E = \nabla_E^{1.0} + \nabla_E^{0.1}$$

$$\Box = \Box_E := \nabla_E^* \nabla_E + \nabla_E \nabla_E^*$$

$$\Box' = \nabla_E^{1.0*} \nabla_E^{1.0} + \nabla_E^{1.0} \nabla_E^{1.0*}$$

$$\Box'' = \nabla_E^{0.1*} \nabla_E^{1.0} + \nabla_E^{0.1} \nabla_E^{0.1*}$$

$$\Box = \Box' + \Box''$$

$E = \Omega^p$. Hodge-decomposition

$\partial^* = -\sqrt{-1}[\Lambda . \partial]$ (?)

(1)
$$\left.\begin{aligned} &[\Lambda . \nabla_E^{0.1}] = -\sqrt{-1}\, \nabla_E^{1.0*} \\ &[\Lambda, \nabla_E^{1.0}] = \sqrt{-1}\, \nabla_E^{0.1*} \\ &[\nabla_E^{0.1*}, L] = \sqrt{-1}\, \nabla_E^{1.0} \\ &[\nabla_E^{1.0*}, L] = -\sqrt{-1}\, \nabla_E^{0.1} \end{aligned}\right\}$$

The proof is the same as we did in Hodge decomposition.

(2) $\Box'' = \Box' + [\sqrt{-1} F, \Lambda]$

$\nabla_E^{1.0}, \nabla_E^{0.1}, L, \Lambda \in \underline{\mathcal{C}^\infty(X, A^{\cdot\cdot}(\mathrm{End}(E)))} \leftarrow$ as a Lie algebra

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\Psi}{A} . B . C$

① $[A, B] = (-1)^{ab} [B . A]$ $\quad a = \deg(A).\ b = \deg(B)$

② $(-1)^{ac} [A, [B . C]] + (-1)^{ba} [B . [C . A]] + (-1)^{cb} [C . [A . B]] = 0$

Jacobi identity

$$\begin{aligned} \Box'' &= [\nabla^{0.1}, \nabla^{0.1*}] \\ &\overset{(1)}{=} [\nabla^{0.1}, -\sqrt{-1}[\Lambda, \nabla^{1.0}]] \\ &= -\sqrt{-1}\left([\Lambda . [\nabla^{1.0} + \nabla^{0.1}]] + [\nabla^{1.0}, [\nabla^{0.1}, \Lambda]]\right) \end{aligned}$$

$$= -\sqrt{-1}\left([\Lambda, F(\nabla)] + \sqrt{-1}\left[\nabla^{1,0}, (\nabla^{1,0})^*\right]\right)$$

$$= \Box' + [\sqrt{-1}\, F(\nabla), \Lambda] \qquad \Box$$

Proof of Kodaira vanishing theorem

show $H^q(X, \underset{\text{positive}}{\Omega^n \otimes \underset{\downarrow}{L}}) = 0 \quad \forall q > 0.$

$(L, h) \longrightarrow \underline{\sqrt{-1}\, F(h_L)} > 0$

$\overset{\shortparallel}{\omega}$ a Kahler form

(X, ω) Kahler mnfd

$g = \omega(\cdot, J\cdot) \qquad \mathrm{Vol}_g = \dfrac{\omega^n}{n!} \qquad *, L, \Lambda$ respect to g

Let $\alpha \in \mathcal{C}^\infty(X, A^{n,q} \otimes L)$, $\bar\partial \alpha = 0$. $\qquad \underline{\alpha \in H^q(X, \Omega^n \otimes L)}$

$\cong$

$\mathcal{H}^{n,q}(L)$

$$\Box''\alpha \overset{\text{Bochner}}{=\!=} \Box'\alpha + [\sqrt{-1}\underset{\overset{\shortparallel}{\omega}}{F(h_L)}, \Lambda]\,\alpha = \Box'\alpha + \underset{(n+q-n)\,\alpha}{\underbrace{[L, \Lambda]\,\alpha}}$$

$$= \Box'\alpha + q\cdot\alpha$$

$\left([L, \Lambda] = H : A^k \to A^{2k}\right.$

$\alpha \to (k-n)\,a.$

Without the loss of generality, we may assume α is a $\Box''$-harmonic form.

$$0 = \int_X (\alpha, \Box''\alpha)_h \frac{\omega^n}{n!}$$

$$= \underset{\overset{\shortparallel}{}}{\underline{\overset{\geq 0}{\int_X (\alpha, \Box'\alpha)\frac{\omega^n}{n!}}}} + q \overset{\geq 0}{\int_X (\alpha, \alpha)_h \frac{\omega^n}{n!}}$$

$$\int\left[(\nabla^{1,0}\alpha, \nabla^{1,0}\alpha) + (\nabla^{1,0*}\alpha, \nabla^{1,0*}\alpha)\right]\frac{\omega^n}{n!}$$

$$\underset{q>0}{\Rightarrow} \quad q\int_X (\alpha, \alpha)_h \frac{\omega^n}{n!} = 0.$$

$$\Rightarrow \alpha = 0 \quad \Longrightarrow \quad H^q(X, \Omega^n \otimes L) = 0$$

$$\{\text{cpt complex mnfd}\} \supsetneq \{\text{cpt. Kahler mnfd}\} \supsetneq \left\{\begin{array}{l}\text{cpt Kahler mnfd } X \\ \text{that admits a positive} \\ \text{line bundle}\end{array}\right\}$$

$X = \bigcap_{1 \leq i \leq k} Z(f_i)$, f_i: <u>homogeneous polynomial</u>

Kodaira embedding $\Big\downarrow$ "1-1"

$$\left\{\begin{array}{l}\text{Smooth submfd } X \subseteq \mathbb{P}^N \\ \text{as an } \left(\begin{array}{l}\text{intersection of} \\ \text{zero loci of} \\ \text{holomogeneous polynomials}\end{array}\right)\end{array}\right\} \xleftrightarrow[\text{theorem}]{\text{Chow's}} \left\{\begin{array}{l}\text{Cpt complex submanifold of a} \\ \text{projective space } X \subseteq \mathbb{CP}^N\end{array}\right\}$$

$$X \subseteq \left(\mathbb{CP}^N, \ \omega_{FS} \in c_1(\mathcal{O}(1)_{\mathbb{CP}^N})\right)$$

$$\omega = \omega_{FS}\big|_X \in c_1(L), \qquad L = \mathcal{O}(1)_{\mathbb{CP}^N}\big|_X$$

$\Longrightarrow$ X is Kahler

admits a positive line bundle.

Hitchin - Kobayashi correspondence.

(Donaldson - Uhlenback - Yau theorem)

Setting : Let (X, ω) be a cpt Kahler mnfd

Let
$$\begin{tikzcd} \mathbb{C}^r \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$
holomorphic vector bundle of rank r.

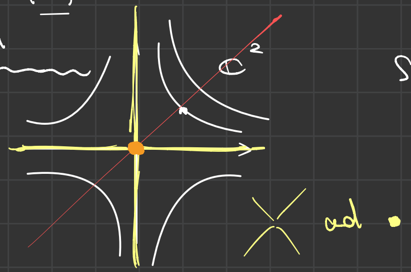
$\tilde{M} = \{(E, h, \nabla^c(h))\}$ $\longleftarrow$ infinity many $\longrightarrow$ study this <u>space</u>.

$\nwarrow$ a topological space

Hausdorff

" $\int_{\tilde{M}} (\underline{\ \ })$ "

e.g.



$SL(2,\mathbb{C}) \supseteq \mathbb{C}^\times \circlearrowright$

$$\lambda > 0 \quad \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$\mathbb{C}^2/\mathbb{C}^\times$

↘ NOT Housdorff

$(z_1, z_2) \in (\mathbb{C}^2 \setminus (\{z_1=0\} \cup \{z_2=0\})) = X$

↑

$\mathbb{C}^2 /\!/ \mathbb{C}^\times = X/\mathbb{C}^*$ Hausdorff "stable points"

"stable" $(E, h, \nabla^c(h))$

Def $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is called slope $\overset{\text{semi-}}{\vee}$stable if For ∀ reflexive coherent subsheaf

stable,

polystable = "$E = \bigoplus_{1 \le i \le k} E_i$ each E_i is stable.

$F \subseteq E$, slope $\mu(F) \le \mu(E)$

$(F \subsetneq E$, slope $\mu(F) < \mu(E).)$

E : $\mathrm{rank}(E) = r$

degree of E : $\deg(E) = \int_X (\Lambda F) \wedge \frac{\omega^{n-1}}{(n-1)!}$

$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $(E, h, F(h))$ ↓ curvature.

$A^{1,1}(\mathrm{End}(E))$

$(\text{trace of } F(h)) \Rightarrow \Lambda F \in A^{1,1}(X)$

slope $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E) \cdot \mathrm{vol}(X)}$, $\mathrm{vol}(X) = \int_X \frac{\omega^n}{n!}$ ← total volume

$F \subseteq E$ a $\underline{\text{refkxive coherent}}$ subsheaf : $\exists\, S \subseteq X$ a closed analytic subset of $\mathrm{codim}_{\mathbb{C}}(S) \ge 2$ $\approx$ "sub holo-vector bundle"

On $X \setminus S$ F is a vector bundle of $\mathrm{rank}(F) = s \le r$

E_X , $E' \subseteq E$ $\qquad E'' = E/E'$ quotient bundle.

$$\begin{tikzcd} E' \arrow[d] & E \arrow[d] \\ \text{Sub bundle} & E \end{tikzcd}$$

$$E(U) = E'(U) \oplus E''(U)$$

(E, h) , $\nabla = d + A$ (w.r.t. a hol. hd frame)

$$A_E = \left(\begin{array}{c|c} A_{E'} & -B \\ \hline B & A_{E''} \end{array} \right)$$

Second fundamental form

$$F(h)_E = \left(\begin{array}{c|c} F(h_{E'}) + A_{E'} \wedge A_{E'} + B \wedge \overset{\geq 0}{\bar{B}^T} & * \\ \hline * & \underset{\leq 0}{F(h_{E''}) - A_{E''} \wedge A_{E''} - B \wedge \bar{B}^T} \end{array} \right)$$

$$F(h_E)\big|_{E'} \geq F(h_{E'})$$

"stable" $\quad \mathbb{C}^\times \curvearrowright \textcircled{E} \xrightarrow{t \to \infty} E' \oplus E''$

$$\textcircled{E} \neq E' \oplus E/E' \qquad E'' = E/E'$$

$$E' \to \begin{pmatrix} \boxed{e^{\lambda_1 t}} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$\underline{\lambda_1 > \lambda_2}$

"not stable"

Hitchin–Kobayashi correspondence.

$$\operatorname{tr}_\omega F(h) = \begin{pmatrix} \overset{\operatorname{tr}_\omega((1,1)\text{-form})}{\bigcirc} & & \\ & & \\ & & \end{pmatrix}_{r \times r} \in \operatorname{End}(E)$$

$\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is $\underline{\text{"polystable"}}$

if and only if

E admits a $\underline{\text{Hermitian–Einstein metric}}$ h

i.e. $\boxed{\operatorname{tr}_\omega F(h) = \mu(E) \cdot \mathrm{Id}_E}$

" 2^{nd}-order elliptic PDE "

Hitchin–Kobayashi correspondence (X, ω) cpt Kahler mnfd

$$\begin{tikzcd} E \arrow[d] \\ X \end{tikzcd}$$

holo vector bundle

$rk(E) = r$

E is polystable iff E admits a Hermitian–Einstein metric H i.e., $tr_\omega F(H) = \mu \cdot Id$

(Hermitian Yang–Mills connection $= \nabla^c(h)$)

$\Lambda^2(End(E))$

i.e. $E = \bigoplus_{1 \le i \le k} E_i$ & E is semistable) → stable

Recall: reflexive coherent subsheaf $F \subseteq E$

$\mu(F) < \mu(E)$

$\Updownarrow$

$E = \bigoplus_{1 \le i \le k} E_i$ ← stable & $\mu(E_i) = \mu(E)$

Proof (Uhlenbeck – Yau)

"$\Leftarrow$" (easier)

"$\Rightarrow$" by "continuity method" to solve 2^{nd}-order elliptic PDE

Denote H as a Hermitian metric on E

$H = \boxed{H_0}\, h$

reference metric

h → "real symmetric matrix"

$u = \log h$

To solve h that satisfies

$$\boxed{K(H) = tr_\omega F(H) - \mu I} = 0$$

$L(h)$

$\varepsilon \in [0,1]$. $\underline{L_\varepsilon(h) = 0}$ $\varepsilon \to 0$.

$\delta L_\varepsilon + \boxed{\text{error}}$

$L_0(h) = L(h)$

$L_1(h)$: "simpler", we can find a solution

Let $U = \{ \varepsilon \in [0,1] : L_\varepsilon = 0$ has a solution $\}$

$U \ne \phi$

$U = [0,1] \Leftrightarrow U$ is open and closed.

$\underline{U \text{ is open}}$ (openness) apply "implicit function theorem" to the "linearization of L_ε"

U is closed (closedness) by a priori estimates

$$\begin{tikzcd}
\underline{\|u\|_{L^2(X)} \text{ is bounded}} \arrow[r, Rightarrow] & \|u\|_{C^0(X)} \leqslant C_0 \arrow[r, Rightarrow] \arrow[dr] & \|u\|_{W^{k,2}(X)} \leqslant C \arrow[d] \arrow[dd, bend right] \\
E \text{ is polystable} \arrow[u, Rightarrow, "\text{(contradiction argument)}"'] & & \|u\|_{C^k(X)} \leqslant C \quad \forall k \in \mathbb{N} \\
& & \left(\sum_{|\alpha| \leq k} \int |\nabla^\alpha u|^2 \omega^n\right)^{1/2}
\end{tikzcd}$$

Hermitian - Einstein $\Longrightarrow$ E is polystable

Let $F \subsetneq E$ reflexive coherent subsheaf
(sub-bundle)

$$\partial H_{\alpha\beta} \wedge \bar{H}_{\alpha\beta} \rightarrow \alpha \in F,\ \beta \in E/F$$

$$\underset{(x)}{\forall x \in X \setminus S} \qquad F(H) = \left(\begin{array}{c|c} F(H_F) + \overset{\geqslant 0}{\underline{2^{nd} \text{ fundamental form}}} & \\ \hline & \\ & \end{array}\right)$$

" $E = F \oplus E/F$ "

$$\frac{\Lambda F(H)}{rk(H)} \geqslant \frac{\Lambda F(H_F)}{rk(H)}$$

$$\Downarrow$$

$$\mu(E) = \int_X \frac{\Lambda F(H)}{rk(H)} \wedge \frac{\omega^{n-1}}{(n-1)!} \geqslant \int_X \frac{\Lambda F(H_F)}{rk(H)} \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu(F)$$

$\Longrightarrow$ E is "semi-stable"

when "=" holds, $\quad H = \left(\begin{array}{c|c} \square & 0 \\ \hline 0 & \square \end{array}\right)$

then $E = F \oplus G$ is an othogonal decomposition

where $\underline{F, G \text{ are sub-bundles of } E}$

$\Longrightarrow \mu(F), \mu(G) \leqslant \mu(E) \Longrightarrow \mu(F) = \mu(G) = \mu(E)$

$$\Longrightarrow \quad E = \bigoplus_{1 \leq i \leq k} E_i \leftarrow \text{stable}$$

polystable $\Longrightarrow$ Hermition - Einstein

$$\nabla_{H_0} = \underbrace{\partial_0}_{\downarrow} + \bar\partial$$

(1,0) part of ∇_{H_0}

Fix reference metric H_0, $H = H_0 \cdot h$

$$K(H) = \mathrm{tr}_\omega \underline{\overset{F(H_0) - \bar\partial(h^{-1}\partial_0 h)}{\overset{\shortparallel}{F(H)}}} - \mu I$$

$\bar\partial(h^{-1}\partial^0 h)$ as 2-form

$\varepsilon \in [0,1]$, $L_\varepsilon(h) = 0$

$$L_\varepsilon(h) = K(H) + \varepsilon \log(h)$$
$$= K(H_0) \underline{- \mathrm{tr}_\omega\left(\bar\partial(h^{-1}\partial_0 h)\right)} + \varepsilon \log h$$
$$\wr\wr$$
$$- \underline{\mathrm{tr}_\omega \bar\partial \partial_0}(\log h)$$
$$\shortparallel$$
$$\Delta_\omega$$
$$\approx -\Delta_\omega(\log h) = f$$

in coordinate

$\bar\partial(h^{-1}\partial^0 h)$ $\quad d\bar z \wedge dz$.

$-\bar\partial(h^{-1}\partial^0 h)$ $\quad dz \wedge d\bar z$

$\underline{(-\Delta_\omega + c)}\, u = f$, $c > 0$

$\downarrow$ has solution. ✓

Surjective

(1) $\varepsilon = 1$, $\exists$ inital solution i.e. $L_1(h) = 0$ has a solution

Let H' be a Hermittion metric, $K(H')$, Let $H_0 = H' \cdot e^{K(H')}$

Then $h = e^{-K(H')}$ is a solution of $L_1(h) = 0$

$$\left(L_1(h) = \underset{\substack{\shortparallel \\ H_0 \cdot h = H'}}{K(H)} + \underset{-K(H')}{\log h} = 0 \nearrow \right)$$

Openness : "linearization" $\rightsquigarrow$ 1^{st}-order variation $\quad$ Fix a frame $\{e_1, \cdots, e_r\}$

$\boxed{\text{Lemma:}}$ $\quad \phi(h) := \sum_{1 \le \alpha \le r} \phi(e^{\lambda\alpha}) e_\alpha \otimes e_\alpha^*$ $\quad$ diaganlized $\Leftarrow$ $\boxed{h = \sum_{1 \le \alpha \le r} e^{\lambda\alpha} e_\alpha \otimes e_\alpha^*}$

$$\delta\phi(h) = \sum_\alpha \phi' e^{\lambda_\alpha} e_\alpha \otimes e_\alpha^* + \sum_{\alpha.\beta} \left(\phi(e^{\lambda_\alpha}) - \phi(e^{\lambda_\beta})\right) \bigcirc\!\!\!\!\!\!\!\!\;\; \delta a_\alpha^\beta \; e_\beta \otimes e_\alpha^*$$

where $\delta e_\alpha = \sum_\beta \delta a_\alpha^\beta e_\beta$

In particular, $\phi(t) = t$

$$\delta h \cdot h^{-1} = \sum_\alpha \delta\lambda_\alpha e_\alpha \otimes e_\alpha^* + \sum_{\alpha.\beta} (1 - e^{\lambda_\beta - \lambda_\alpha}) \delta a_\alpha^\beta e_\beta \otimes e_\alpha^*$$

$\phi(t) = \log t$

$$\delta \log h = \sum_\alpha \delta\lambda_\alpha e_\alpha \otimes e_\alpha^* + \sum_{\alpha.\beta}$$

$\Longrightarrow$ (1) normalization. $\underset{e^{\lambda_1+\cdots+\lambda_r}}{\underset{\|}{\delta \log(\det(h))}} = \underset{\sum_\alpha (\delta h h^{-1})_\alpha}{\underset{\|}{<\delta h h^{-1}, I>}} = 0$ $\quad (\log(\det(h)) \equiv \text{const})$

We can compute $\underline{\delta L_\varepsilon(h)} = \delta(K(\cancel{H_0}) - \mathrm{tr}_\omega \bar\partial(h^{-1}\partial_0 h) + \varepsilon \log h)$,

and do integration by parts.

$$\int_X <\delta L_\varepsilon(h), \delta h h^{-1}> \geq C(\overset{\|\log h\|_{C^0}}{\overset{\|}{m}}, \varepsilon)\left(\underline{\int_X |\partial^0(\delta h h^{-1})|^2 + \int_X |\delta h h^{-1}|^2}\right)$$

$|\delta h \cdot h^{-1}|_{W^{2,2}}$

$\Longrightarrow$ $\delta L_\varepsilon : W_0^{2,2} \longrightarrow L_0^2$ is surjective

Then by implicit function theorem, we have openness in "$\underline{\text{weak space}}$"

(with the a priori estimate $(C^0, C^k\text{-estimates}) \Rightarrow$ "openness")

"Closedness" (a perior $\underline{\text{estimates}}$)

$\downarrow$

$C^0, C^k, W^{k,2}$

Let $u = \log h$,

$$< \partial^0 u, \partial^0 h h^{-1}> = \sum_\alpha |\partial\lambda_\alpha|^2 + \sum_{\alpha,\beta} |a_\alpha^\beta|^2 \underbrace{(\lambda_\alpha - \lambda_\beta)(1 - e^{\lambda_\beta - \lambda_\alpha})}_{\geq 0} \geq \sum_\alpha |\partial^0 \lambda_\alpha|^2$$

$\|$

$<\partial^0 u, \partial^0 u> = |\partial^0 u|^2$

"$\partial^0 = \nabla_{H_0}^{1,0}$"

$$<u, \bar\partial(h^{-1}\partial^0 h)> = \partial\left(\sum_\alpha \lambda_\alpha (\bar\partial\lambda_\alpha)\right) - \sum_\alpha |\partial\lambda_\alpha|^2$$

$$= \frac{1}{2}\Delta_\omega\left(\sum_\alpha |\lambda_\alpha|^2\right) = \frac{1}{2}\Delta_\omega(|u|^2)$$

$$\underset{K(H_0) + \mathrm{tr}_\omega(\bar\partial h^{-1}\partial^0 h) + \varepsilon \cdot u}{\overset{0}{\overset{\|}{0 = <\underset{\|}{L_\varepsilon(h)}, u>}}} \geq \underset{< C|u|}{\underbrace{<K(H_0), u>}} - \frac{1}{2}\Delta_\omega |u|^2 + \varepsilon \cdot |u|^2$$

$$\Longrightarrow \Delta_\omega |u|^2 \geq 2\varepsilon |u|^2 - C|u| \geq -C|u|.$$

$\underline{\underline{C^0\text{-estimate}}}$ (Moser iteration) $\displaystyle\int_X -|u|^{p-1}\Delta|u|^2 \omega^n \leq \int_X C|u|^p \omega^n$

(depends $\|u\|_{L^2}$)

$$(p-1)\int_X |u|^{p-2}|\nabla|u||^2 = \frac{4(p-1)}{(p+1)^2}\int_X |\nabla |u|^{\frac{p+1}{2}}|^2$$

$$\Longrightarrow \int_X |\nabla|u|^{\frac{p+1}{2}}|^2 \leq C\cdot(p+1)\int_X (|u|^{p/2})^{\underset{\frac{p+1}{p}}{\underset{\downarrow}{2}}} \cdot 1 \leq C'(p+1)\left(\int_X \underline{|u|^{p+1}}\right)^{\frac{p}{p+1}}$$

$\downarrow$ Hölder.

$\to (|u|^{\frac{p+1}{2}})^2$

by Sobolev inequality : $\chi = \frac{2n}{2n-2} = \frac{n}{n-1}$

$$\left(\int \left(|u|^{\frac{p+1}{2}} \right)^{\cancel{\frac{2n}{n-1}}} \right)^{\frac{n-1}{n}} \leq C'(p+1) \left(\int_X |u|^{p+1} \right)^{\frac{p}{p+1}}$$

$$\left(\int |u|^{(p+1)\chi} \right)^{1/\chi} \leq C'(p+1) \left(\int_X |u|^{p+1} \right)^{\frac{p}{p+1}} \leq C'(p+1) \left(\int_X |u|^{p+1} \right)$$

$$\Rightarrow \|u\|_{L^{(p+1)\chi}} \leq \left(C(p+1) \right)^{\frac{1}{p+1}} \|u\|_{L^{p+1}}$$

$$\boxed{\begin{array}{c}\text{start from } p=1 \\ \text{do iteration}\end{array}}$$

$$\Rightarrow \log\Big(\underset{\underset{\|u\|_{C^0}}{\|}}{\|u\|_{L^\infty}} \Big) \leq \log\left(\|u\|_{L^2} \right) + \underline{\frac{1}{2}\log(2C) + \frac{1}{2\chi}\log(2\chi C) + \frac{1}{2\chi^2}\log(2\chi^2 \cdot C) + \cdots}$$

$$\underbrace{+ \frac{1}{2\chi^n}\log(2\chi^n C) + \cdots}_{\leq C} \Big) \quad \chi > 1 . \quad \downarrow \text{ convergent!}$$

$$\Rightarrow \|u\|_{C^0} \leq C \|u\|_{L^2}$$

$W^{k,2}$ estimates of u. $\underset{\underset{\delta \log h \,\approx\, \delta h h^{-1}}{\|}}{\underline{\delta u}} \iff \Phi = h^{-1/2} \delta h h^{-1/2}$

$$\Delta |\Phi|^2 \geq \varepsilon |\delta h h^{-1}|^2 + \langle \log h, \Phi \rangle \qquad \|u\|_{L^2(X)} \leq m$$

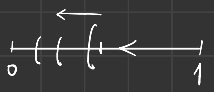
(integration by parts + estimates) $\Rightarrow \|\delta u\|_{W^{k,2}} \leq C(m)$

$\Rightarrow \|u\|_{W^{k,2}} \leq C(m)$

C^k- estimate (higher order estimate)

$$\|u\|_{C^k} \leq C(m, \|H_0\|_{C^k}) \quad \text{if } n \leq \delta_0$$

Assume $\|u\|_{L^2} \leq m$ uniformly



$\Rightarrow \|u\|_{W^{k,2}} \leq C(m)$

$\Rightarrow$ closedness together with openness $\Rightarrow L_0(h) = 0$ has a solution

Polystable $\Longrightarrow$ $\|u\|_{L^2} \le m$ uniformly for $\forall\, \varepsilon \in [0,1]$

"stable"

By contradiction argument

Assume $\varepsilon_j \searrow$ as $j \to +\infty$. $\|u\|_{L^2} \longrightarrow +\infty$

$$\sup_{x\in X}\sup_{1\le\alpha\le r}|\lambda_\alpha| \longrightarrow +\infty$$

$\log \rho_j$

$$\sup_{x\in X}\sup_{1\le\alpha\le r}\lambda_\alpha \xrightarrow{j\to+\infty} +\infty$$

$$\inf_{x\in X}\inf_{1\le\alpha\le r}\lambda_\alpha \xrightarrow{j\to+\infty} +\infty$$

$$u = \log h = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r\end{pmatrix}, \qquad \|u\|_{L^2}^2 \sim \sum_\alpha \int |\lambda_\alpha|^2$$

normalization $\Lambda \log h = 0$ $\quad \sum_{1\le\alpha\le r}\lambda_\alpha = 0 \longrightarrow$

$$h_j := \rho_j\, h(\varepsilon_j) \qquad h_j = \begin{pmatrix} e^{\lambda_1+\log\rho_j} & & \\ & \ddots & \\ & & e^{\lambda_r+\log\rho_j}\end{pmatrix} \qquad \lambda_\alpha + \log\rho_j \le 0$$

$\|h_j\|_{L^2} \le C$ uniformly

$h_j \xrightarrow{W^{1,p}} h_\infty$ for $\forall\, p>1$

$$\| \qquad \begin{pmatrix} e^{\lambda_1'} & & \\ & \ddots & \\ & & e^{\lambda_r'}\end{pmatrix}$$

$$\inf_x\inf_\alpha \lambda_\alpha \xrightarrow{j\to+\infty} -\infty$$

$$\begin{pmatrix} \boxed{\begin{matrix} e^{\lambda_1'} & \\ & e^{\lambda_2'}\end{matrix}} & \\ & \textcircled{0}\end{pmatrix} \qquad E \supseteq E'$$

$\textcircled{0} \to$ "sub-bundle" E'

$E' =$ " where $e^{\lambda_\alpha} = 0$ " is a holo "sub-bundle" of E

↳ (Uhlenbeck-Yau. Siu)

$E' \subset E$

$\mu(E') > \mu(E)$

by 'equation'.

Contradicts with E is stable

$\Longrightarrow$ $\|u\|_{L^2}$ is uniformly bounded.

□